CHAPTER 7

Centroid of Plane Figures

Centroid is an important property of the cross-section of a member which is required frequently in the analysis of many engineering problems. In this chapter the meaning of centre of gravity and centroid of plane figures is explained. Centroid of triangle, semicircle, quadrant of a circle and sector of a circle using method of integration is presented. Method of locating centroid of simple built up sections is illustrated by solving numerical problems.

7.1 CENTRE OF GRAVITY

Consider a suspended body as shown in Fig. 7.1. The weights of various parts of this body are acting vertically downward. The only upward force is the force in the string. To satisfy the equilibrium condition the resultant weight of the body W must act along the line of the string (1)-(1). Now, if the position is changed and the body is suspended again, it will reach equilibrium in a particular position. Let the line of action of resultant weight be (2)-(2) intersecting line (1)-(1) at G. It is found that if the body is suspended in any other position, the line of action of resultant weight W passes through G. This point is called the centre of gravity. Thus, *centre of gravity can be defined as the point through which resultant of force of gravity (weight) of the body acts.*

7.2 CENTRE OF GRAVITY OF FLAT PLATE

Consider a flat plate of thickness t as shown in Fig. 7.2. Let W_i be the weight of any elemental portion acting at a point (x_i, y_i). Let W be the total weight of the plate acting at the point $(\bar{x}, \bar{y})$. According to definition of centre of gravity, the point $(\bar{x}, \bar{y})$ is the centre of gravity. Now,

$$\text{Total weight } W = \sum W_i \qquad \text{Eqn. (7.1)}$$

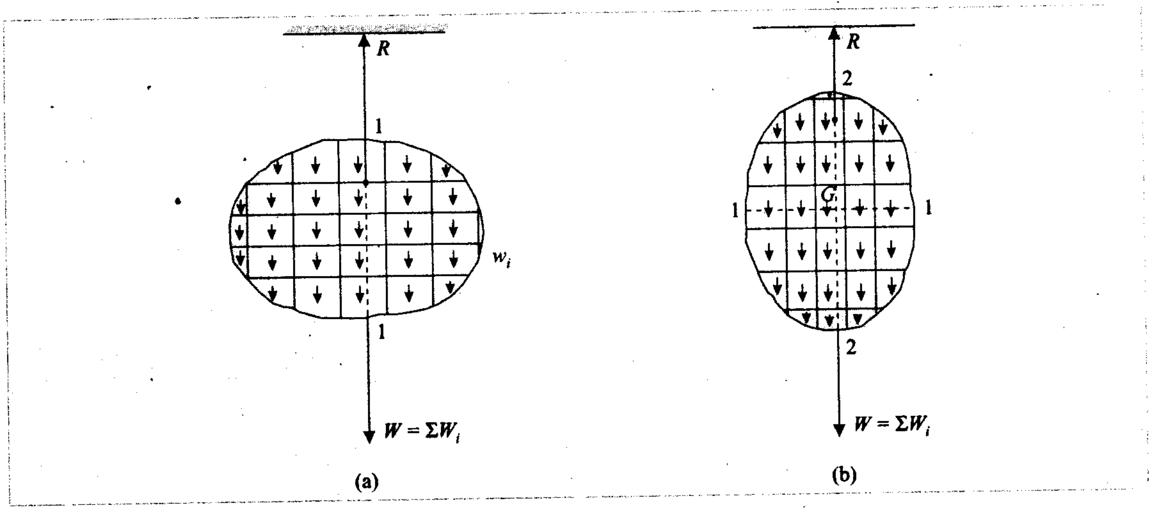


Fig. 7.1

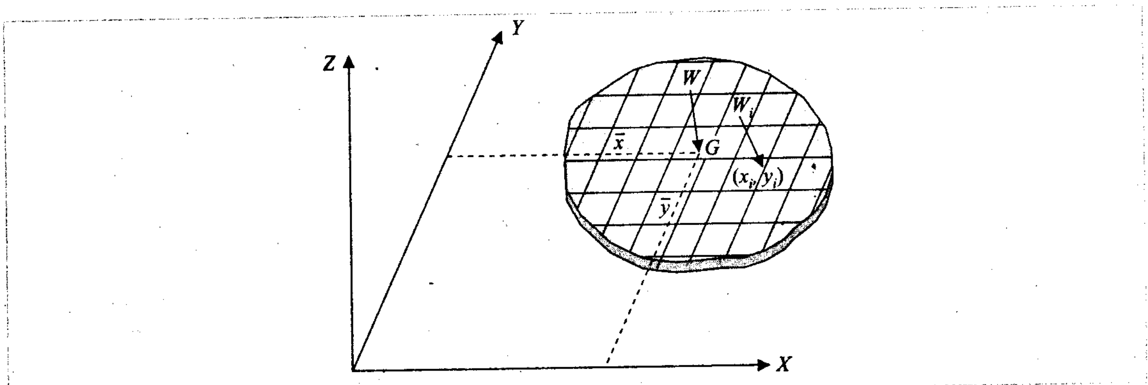


Fig. 7.2

Taking moment about x-axis and equating moment of resultant to moment of component forces, we get

$$W\bar{y} = W_1y_1 + W_2y_2 + W_3y_3 + \dots$$
$$= \Sigma W_iy_i$$

$$\therefore \quad \bar{y} = \frac{\Sigma W_iy_i}{W} \qquad \text{Eqn. (7.2)}$$

Similary, taking moment about y-axis we get,

$$W\bar{x} = W_1x_1 + W_2x_2 + W_3x_3 + \dots$$
$$= \Sigma W_ix_i$$

$$\bar{x} = \frac{\Sigma W_ix_i}{W} \qquad \text{Eqn. (7.3)}$$

7.3 CENTROID

Let A_i be the area of the ith element of plate of uniform thickness in the plate. If γ is the unit weight of the material of plate and t its uniform thickness, then

$$W_i = \gamma A_i t$$

∴ Total weight,

$$W = \Sigma \gamma A_i t = \gamma t \Sigma A_i$$

$$= \gamma t A \qquad \text{Eqn. (7.4)}$$

where, $A = \Sigma A_i$ is total area.

From equations 7.2 and 7.3 we get

$$\bar{y} = \frac{\Sigma A_i \gamma t y_i}{\gamma t A} = \frac{\Sigma A_i y_i}{A} \qquad \text{Eqn. (7.5)}$$

and

$$\bar{x} = \frac{\Sigma A_i \gamma t x_i}{\gamma t A} = \frac{\Sigma A_i x_i}{A} \qquad \text{Eqn. (7.6)}$$

since γ and t are constants.

The terms $\Sigma A_i y_i$ and $\Sigma A_i x_i$ may be considered as moment of area about x-axis and y-axis. Thus, the distance of the centre of gravity of plate of uniform thickness from an axis can be located by dividing moment of area about that axis by the total area. It will not depend upon the magnitude of thickness of plate and unit weight of material. If the thickness reduces to infinitesimal, then the plate reduces to an area. Still the expressions for finding centre of gravity of the area are same as Eqn. 7.5 and 7.6. Centre of gravity is a misnomer for the area. It is to be called as **centroid**. It may be noted that since the moment of area about an axis divided by total area gives the distance of centroid from that axis, the moment of area is zero about any centroidal axis.

7.4 DIFFERENCE BETWEEN CENTRE OF GRAVITY AND CENTROID

From the above discussion we can draw the following differences between centre of gravity and centroid:

(1) The term centre of gravity applies to bodies with mass and weight, and centroid applies to plane areas.

(2) Centre of gravity of a body is a point through which the resultant gravitational force (weight) acts for any orientation of the body whereas centroid is a point in a plane area such that the moment of area about any axis through that point is zero.

7.5 USE OF AXIS OF SYMMETRY

Centroid of an area lies on the axis of symmetry if it exists. This is a useful theorem to locate the centroid of an area. This theorem can be proved as follows:

Consider the area shown in Fig. 7.3. In this figure $y - y$ is the axis of symmetry. From Eqn. 7.6, the distance of centroid from this axis is given by:

$$\frac{\sum A_i x_i}{A}$$

Consider the two elemental areas shown in Fig. 7.3, which are equal in size and are equidistant from the axis, but on either side. Now the sum of moments of these areas cancel each other since the areas and distances are the same, but signs of distances are opposite. Similarly, we can go on considering an area on one side of symmetric axis and corresponding image area on the other side, and prove that total moments of area ($\Sigma A_i x_i$) about the symmetric axis is zero. Hence the distance of centroid from the symmetric axis is zero, i.e., centroid always lies on symmetric axis.

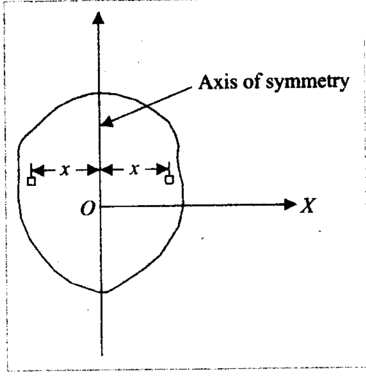


Fig. 7.3

Making use of the symmetry we can conclude that:

(1) Centroid of a circle is its centre (Fig. 7.4);

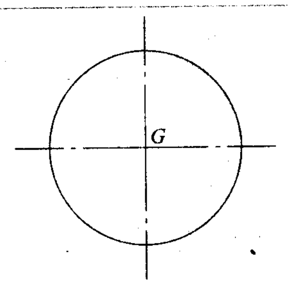


Fig. 7.4

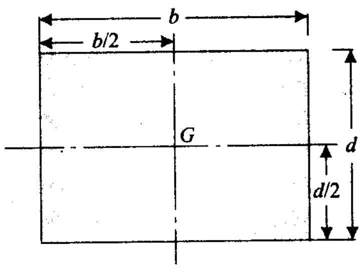


Fig. 7.5

(2) Centroid of a rectangle of sides b and d is at a distance $\frac{b}{2}$ and $\frac{d}{2}$ from any corner (Fig. 7.5).

7.6 DETERMINATION OF CENTROID OF SIMPLE FIGURES FROM METHOD OF INTEGRATION

For simple figures like triangle and semicircle, we can write general expression for the elemental area and its distance from an axis. Then equations 7.5 and 7.6 reduce to:

$$\bar{y} = \frac{\int y dA}{A} \qquad \text{Eqn. (7.7)}$$

$$\bar{x} = \frac{\int x dA}{A} \qquad \text{Eqn. (7.8)}$$

The location of the centroid using the above equations may be considered as finding centroid from method of integration. Now, let us find centroid of some standard figures from first principles.

Centroid of a Triangle — Consider the triangle ABC of base width b and height h as shown in Fig. 7.6. Let us locate the distance of centroid from the base. Let b_1 be the width of elemental strip of thickness dy at a distance y from the base. Since ΔAEF and ΔABC are similar triangles, we can write:

$$\frac{b_1}{b} = \frac{h-y}{h}$$

$$b_1 = \left(\frac{h-y}{h}\right)b = \left(1-\frac{y}{h}\right)b$$

$\therefore$ Area of the element $= dA = b_1 dy$

$$= \left(1-\frac{y}{h}\right)b\,dy$$

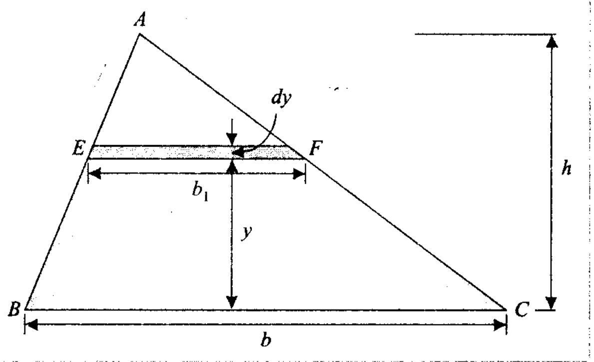


Fig. 7.6

Area of the triangle $A = \frac{1}{2}bh$

$\therefore$ From Eqn. 7.7

$$\bar{y} = \frac{\text{Moment of area}}{\text{Total area}} = \frac{\int y\,dA}{A}$$

Now,

$$\int y\,dA = \int_0^h y\left(1-\frac{y}{h}\right)b\,dy$$

$$= \int_0^h \left(y-\frac{y^2}{h}\right)b\,dy$$

$$= b\left[\frac{y^2}{2}-\frac{y^3}{3h}\right]_0^h$$

$$= \frac{bh^2}{6}$$

$$\bar{y} = \frac{\int y\,dA}{A} = \frac{bh^2}{6} \times \frac{1}{\frac{1}{2}bh}$$

$$\therefore \qquad \bar{y} = \frac{h}{3}$$

Thus the centroid of a triangle is at a distance $\frac{h}{3}$ from the base (or $\frac{2h}{3}$ from the apex) of the triangle where h is the height of the triangle.

Centroid of a Semicircle – Consider the semicircle of radius R as shown in Fig. 7.7. Due to symmetry centroid must lie on y-axis. Let its distance from diametral axis be $\bar{y}$. To find $\bar{y}$, consider an element at a distance r from the centre O of the semicircle, radial width being dr and bound by radii at θ and $\theta + d\theta$.

The elemental area may be treated as a rectangle of sides $rd\theta$ and dr. Hence

$$\text{Area of element} = rd\theta\ dr.$$

Its moment about diametral axis x is given by:

$$rd\theta \times dr \times r \sin\theta$$
$$= r^2 \sin\theta\ drd\theta$$

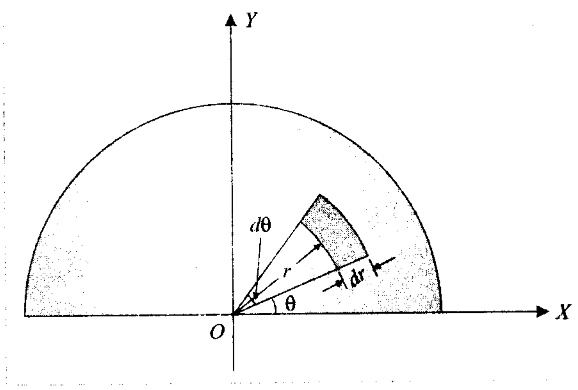


Fig. 7.7

∴ Total moment of area about diametral axis,

$$= \int_0^{\pi}\int_0^{R} r^2 \sin\theta\ dr\ d\theta$$

$$= \int_0^{\pi}\left[\frac{r^3}{3}\right]_0^R \sin\theta\ d\theta$$

$$= \frac{R^3}{3}[-\cos\theta]_0^{\pi}$$

$$= \frac{R^3}{3}[1+1] = \frac{2R^3}{3}$$

Area of semicircle $A = \frac{1}{2}\pi R^2$

$$\therefore \qquad \bar{y} = \frac{\text{Moment of area}}{\text{Total area}} = \frac{\frac{2R^3}{3}}{\frac{1}{2}\pi R^2}$$

$$= \frac{4R}{3\pi}$$

Thus, the centroid of the circle is at a distance $\frac{4R}{3\pi}$ from the diametral axis.

Centroid of Quadrant of a Circle

Its moment about diametral x-axis is given by $\int_0^{\pi/2}\int_0^{R} r^2 \sin\theta\ dr\ d\theta$.

[**Note:** *change from that of a semicircle is only in the limit of integration for θ*].

∴ Total moment of area about diametral axis

$$= \frac{R^3}{3}.$$

Area of quadrant of circle $= \frac{1}{4}\pi R^2$

∴ $\bar{y} = \frac{4R}{3\pi}$, as in the previous case. Similarly by taking moment of area about y-axis it can be shown that

$$\bar{x} = \frac{4R}{3\pi}$$

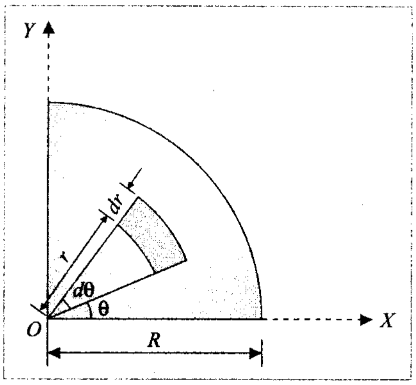


Fig. 7.8

Centroid of Sector of a Circle — Consider the sector of a circle of angle 2 α as shown in Fig. 7.9. Due to symmetry, centroid lies on x axis. To find its distance from the centre O, consider the elemental area shown.

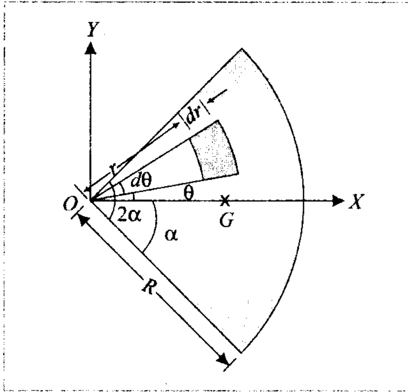


Fig. 7.9

Area of the element $= r\,d\theta\,dr$

Its moment about y-axis

$$= rd\theta \times dr \times r\cos\theta$$

$$= r^2 \cos\theta \, dr \, d\theta$$

∴ Total moment of area about y-axis

$$= \int_{-\alpha}^{\alpha}\int_{0}^{R} r^2 \cos\theta \, dr \, d\theta$$

$$= \left[\frac{r^3}{3}\right]_0^R [\sin\theta]_{-\alpha}^{\alpha}$$

$$= \frac{R^3}{3} 2\sin\alpha$$

Total area of the sector

$$= \int_{-\alpha}^{\alpha}\int_{0}^{R} r\,dr\,d\theta$$

$$= \int_{-\alpha}^{\alpha}\left[\frac{r^2}{2}\right]_0^R d\theta$$

$$= \frac{R^2}{2}[\theta]_{-\alpha}^{\alpha}$$

$$= R^2\alpha$$

∴ The distance of centroid from centre O

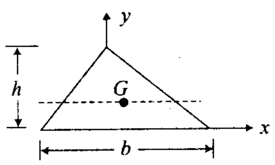
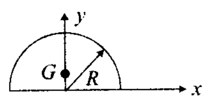
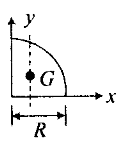
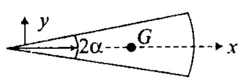
$$= \frac{\text{Moment of area about } y \text{ axis}}{\text{Area of the figure}}$$

$$= \frac{\frac{2R^3}{3}\sin\alpha}{R^2\alpha} = \frac{2R}{3\alpha}\sin\alpha$$

7.7 CENTROID OF SIMPLE BUILT UP SECTIONS

So far, the discussion was confined to locating the centroid of simple figures like rectangle, triangle, circle, semicircle, etc. In engineering practice, use of sections which are built up of many simple sections is very common. Such sections may be called as built-up sections or composite sections. To locate the centroid of composite sections, one need not go for the first principle (method of integration). The given composite section can be split into suitable number of simple figures and then the centroid of each simple figure can be found by inspection or using the standard formulae listed in Table 7.1. Assuming the area of the simple figure as concentrated at its centoid, its moment about an axis can be found by multiplying the area with distance of its centroid from the reference axis. After determining moment of each area about reference axis, the distance of centroid from the axis is obtained by dividing total moment of area by total area of the composite section.

Table 7.1 Centroid of Some Common Figures

Shape	*Figure*	$\bar{x}$	$\bar{y}$	*Area*
Triangle		–	$\frac{h}{3}$	$\frac{bh}{2}$
Semicircle		0	$\frac{4R}{3\pi}$	$\frac{\pi R^2}{2}$
Quarter circle		$\frac{4R}{3\pi}$	$\frac{4R}{3\pi}$	$\frac{\pi R^2}{4}$
Sector of a circle		$\frac{2R}{3\alpha}\sin\alpha$	0	αR^2

Example 7.1 Locate the centroid of the *T*-section shown in the Fig. 7.10.

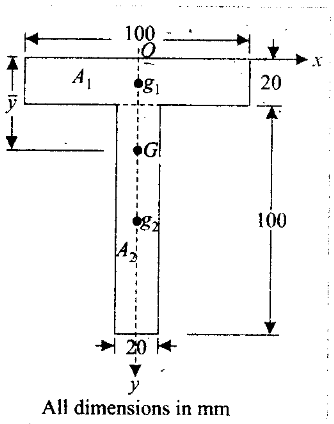


Fig. 7.10

Solution.

Selecting the axis as shown in Fig. 7.10, we can say due to symmetry centroid lies on y axis, i.e., $\bar{x} = 0$.

Now the given *T*-section may be divided into two rectangles A_1 and A_2 each of size 100×20 and 20×100. The centroid of A_1 and A_2 are g_1 (0, 10) and g_2 (0, 70), respectively.

$\therefore$ The distance of centroid from top is given by:

$$\bar{y} = \frac{100 \times 20 \times 10 + 20 \times 100 \times 70}{100 \times 20 + 20 \times 100}$$

$$= 40 \text{ mm}$$

Hence, **centroid of T-section is on the symmetric axis at a distance 40 mm from the top.** **Ans.**

Example 7.2 Find the centroid of the unequal angle $150 \times 200 \times 12$ mm, shown in Fig. 7.11.

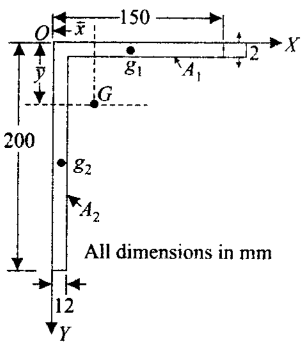


Fig. 7.11

Solution.

The given composite figure can be divided into two rectangles:

$$A_1 = 150 \times 12 = 1800 \text{ mm}^2$$

$$A_2 = (200 - 12) \times 12 = 2256 \text{ mm}^2$$

Total area $A = A_1 + A_2 = 4056 \text{ mm}^2$

Selecting the reference axes x and y as shown in Fig. 7.11, the centroid of A_1 is g_1 (75, 6) and that of A_2 is:

$$g_2 \left[6, 12 + \frac{1}{2}(200 - 12)\right]$$

i.e., g_2 (6, 106).

$$\therefore \quad \bar{x} = \frac{\text{Moment about } y \text{ axis}}{\text{Total area}}$$

$$= \frac{A_1 x_1 + A_2 x_2}{A}$$

$$= \frac{1800 \times 75 + 2256 \times 6}{4056}$$

$$= 36.6 \text{ mm}$$

$$\bar{y} = \frac{\text{Moment about } x \text{ axis}}{\text{Total area}}$$

$$= \frac{A_1 y_1 + A_2 y_2}{A}$$

$$= \frac{1800 \times 6 + 2256 \times 106}{4056}$$

$$= 61.6 \text{ mm}$$

Thus, the centroid is at $\bar{x}$ = **36.6 mm** and $\bar{y}$ = **61.6 mm** as shown in the figure. **Ans.**

Example 7.3 Locate the centroid of the I-section shown in Fig. 7.12.

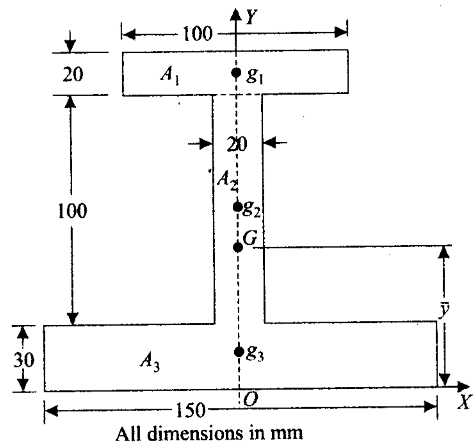


Fig. 7.12

Solution.

Selecting the co-ordinate system as shown in Fig. 7.12, due to symmetry centroid must lie on y-axis,

i.e., $\bar{x} = 0$

Now, the composite section may be split into three rectangles.

$$A_1 = 100 \times 20 = 2000 \text{ mm}^2.$$

Centroid of A_1 from the x-axis is:

$$y_1 = 30 + 100 + \frac{20}{2} = 140 \text{ mm}$$

Similarly $$A_2 = 100 \times 20 = 2000 \text{ mm}^2$$

$$y_2 = 30 + \frac{100}{2} = 80 \text{ mm}$$

$$A_3 = 150 \times 30 = 4500 \text{ mm}^2, \text{ and}$$

$$y_3 = \frac{30}{2} = 15 \text{ mm}$$

$\therefore$ $$\bar{Y} = \frac{A_1Y_1 + A_2Y_2 + A_3Y_3}{A}$$

$$= \frac{2000 \times 140 + 2000 \times 80 + 4500 \times 15}{2000 + 2000 + 4500}$$

$$= 59.7 \text{ mm}$$

Thus, **the centroid is on the symmetric axis at a distance 59.7 mm from the bottom** as shown in Fig. 7.12. **Ans.**

Example 7.4 Determine the centroid of the section of the concrete dam shown in Fig. 7.13.

Solution.

Let the axis be selected as shown in Fig. 7.13. Note that it is convenient to take axes in such a way that the centroids of all simple figures are having positive coordinates. If coordinate of any simple figure comes out to be negative, one should be careful in assigning the sign for moment of area of that figure.

The composite figure can be conveniently divided into two triangles and two rectangles, as shown in Fig. 7.13.

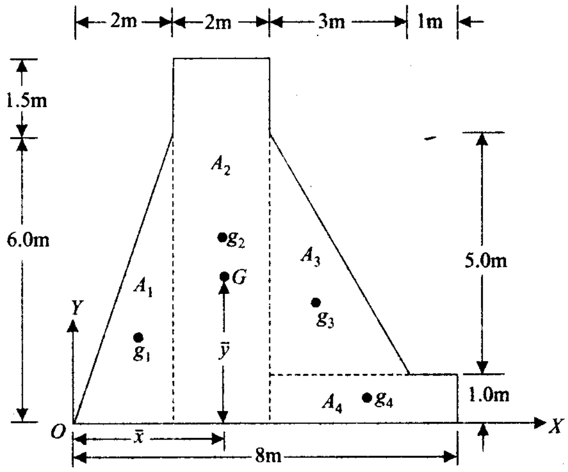


Fig. 7.13

Now, $$A_1 = \frac{1}{2} \times 2 \times 6 = 6 \text{ m}^2$$

$$A_2 = 2 \times 7.5 = 15 \text{ m}^2$$

$$A_3 = \frac{1}{2} \times 3 \times 5 = 7.5 \text{ m}^2$$

$$A_4 = 1 \times 4 = 4 \text{ m}^2$$

$$A = \text{total area} = 32.5 \text{ m}^2$$

Centroides of simple figures are:

$$x_1 = \frac{2}{3} \times 2 = \frac{4}{3} \text{ m}$$

$$y_1 = \frac{1}{3} \times 6 = 2 \text{ m}$$

$$x_2 = 2 + 1 = 3 \text{ m}$$

$$y_2 = \frac{7.5}{2} = 3.75 \text{ m}$$

$$x_3 = 2 + 2 + \frac{1}{3} \times 3 = 5 \text{ m}$$

$$y_3 = 1 + \frac{1}{3} \times 5 = \frac{8}{3} \text{ m}$$

$$x_4 = 4 + \frac{4}{2} = 6 \text{ m}$$

$$y_4 = 0.5 \text{ m}$$

$$\bar{x} = \frac{A_1x_1 + A_2x_2 + A_3x_3 + A_4x_4}{A}$$

$$= \frac{6 \times \frac{4}{3} + 15 \times 3 + 7.5 \times 5 + 4 \times 6}{32.5}$$

$$= 3.523 \text{ m}$$

$$\bar{y} = \frac{A_1y_1 + A_2y_2 + A_3y_3 + A_4y_4}{A}$$

$$= \frac{6 \times 2 + 15 \times 3.75 + 7.5 \times \frac{8}{3} + 4 \times 0.5}{32.5}$$

$$= 2.777 \text{ m}$$

The centroid is at $\bar{x} \doteq$ **3.523 m** **Ans.**

and $\bar{y}$ = **2.777 m** **Ans.**

Example 7.5 Determine the centroid of the area shown in Fig. 7.14 with respect to the axes shown.

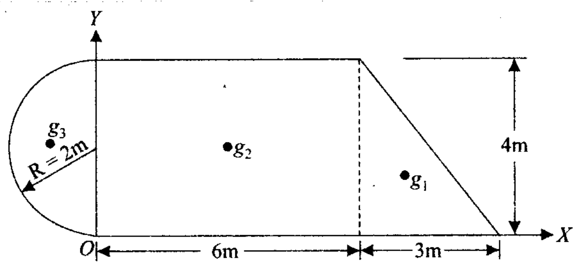


Fig. 7.14

Solution.

The composite section is divided into three simple figures, a triangle, a rectangle and a semicircle.

Now, area of triangle $A_1 = \frac{1}{2} \times 3 \times 4 = 6 \text{ m}^2$

Area of rectangle $A_2 = 6 \times 4 = 24 \text{ m}^2$

Area of semicircle $A_3 = \frac{1}{2} \times \pi \times 2^2 = 6.283 \text{ m}^2$

∴ Total area $A = 36.283 \text{ m}^2$

The coordinates of centroids of these simple figures are:

$$x_1 = 6 + \frac{1}{3} \times 3 = 7 \text{ m}$$

$$y_1 = \frac{4}{3} \text{m}$$

$$x_2 = 3 \text{ m}$$

$$y_2 = 2 \text{ m}$$

$$x_3 = \frac{-4R}{3\pi} = -\frac{4 \times 2}{3\pi} = -0.849 \text{ m}$$

$$y_3 = 2 \text{ m}$$

(Note carefully the sign of x_3).

$$\bar{x} = \frac{A_1x_1 + A_2x_2 + A_3x_3}{A}$$

$$= \frac{6 \times 7 + 24 \times 3 + 6.283 \times (-0.849)}{36.283}$$

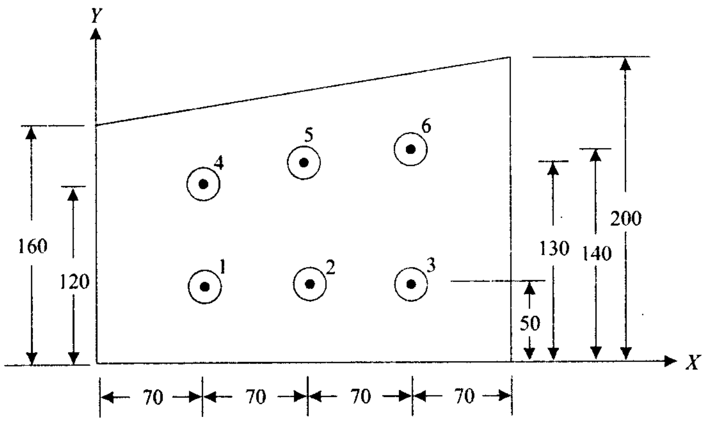
i.e., $\bar{x}$ = **2.995 m** **Ans.**

$$\bar{y} = \frac{A_1y_1 + A_2y_2 + A_3y_3}{A}$$

$$= \frac{\frac{6 \times 4}{3} + 24 \times 2 + 6.283 \times 2}{36.283}$$

i.e., $\bar{y}$ = **1.890 m** **Ans.**

Example 7.6 In a gusset plate, there are six rivet holes of 21.5 mm diameter as shown in Fig. 7.15. Find the position of the centroid of the gusset plate.



Solution.

The composite area is equal to a rectangle of size 160 × 280 mm plus a triangle of size 280 mm base width and 40 mm height and minus areas of six holes. In this case also the Eqns. 7.5 and 7.6

can be used for locating centroid by treating area of holes as negative. The areas of simple figures and their certroids are as shown in Table 7.2 given below:

Table 7.2

Figure	*Area in mm²*	x_i *in mm*	y_i *in mm*
Rectangle	$160 \times 280 = 44{,}800$	140	80
Triangle	$\frac{1}{2} \times 280 \times 40 = 5600$	$\frac{560}{3}$	$160 + \frac{40}{3} = 173.33$
1st hole	$\frac{-\pi \times 21.5^2}{4} = -363.05$	70	50
2nd hole	−363.05	140	50
3rd hole	−363.05	210	50
4th hole	−363.05	70	120
5th hole	−363.05	140	130
6th hole	−363.05	210	140

$$\therefore \quad A = \sum A_i = 48{,}221.70$$

$$\therefore \sum A_i x_i = 44{,}800 \times 140 + 5600 \times \frac{560}{3} - 363.05\,(70 + 140 + 210 + 70 + 140 + 210)$$

$$= 70{,}12{,}371.3 \text{ mm}^3$$

$$\bar{x} = \frac{\sum A_i x_i}{A} = 145.4 \text{ mm}$$

$$\sum A_i y_i = 44{,}800 \times 80 + 5600 \times 173.33 - 363.05\,(50 \times 3 + 120 + 130 + 140)$$

$$= 43{,}58{,}601 \text{ mm}^3$$

$$\bar{y} = \frac{\sum A_i y_i}{A} = \frac{43{,}58{,}601}{48221.70} = 90.4 \text{ mm}$$

Thus, the coordinates of centroid of composite figure are given by:

$\bar{x}$ **= 145.4 mm** **Ans.**

$\bar{y}$ **= 90.4 mm** **Ans.**

Example 7.7 Determine the coordinates x_c and y_c of the centre of a 100 mm diameter circular hole cut in a thin plate so that this point will be the centroid of the remaining shaded area shown in Fig. 7.16 (All dimensions are in mm).

Solution.

If x_c and y_c are the coordinates of the centre of the circle, centroid also must have the Coordinates x_c and y_c as per the condition laid down in the problem. The shaded area may be considered as a rectangle of size 200 mm × 150 mm *minus* a triangle of sides 100 mm × 75 mm and a circle of diameter 100 mm.

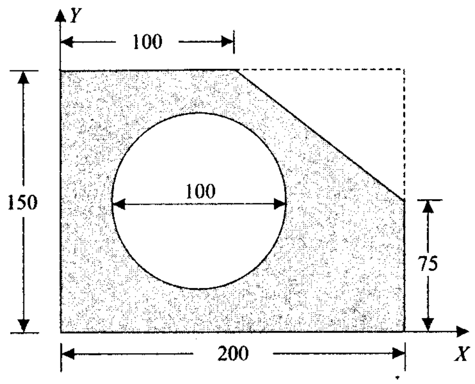


Fig. 7.16

$\therefore$ $$\text{Total area} = 200 \times 150 - \frac{1}{2} \times 100 \times 75 - \left(\frac{\pi}{4}\right)100^2$$

$$= 18{,}396 \text{ mm}^2$$

$$\bar{x} = x_c = \frac{200 \times 150 \times 100 - \frac{1}{2} \times 100 \times 75 \times \left[200 - \left(\frac{100}{3}\right)\right] - \frac{\pi}{4} \times 100^2 \times x_c}{18{,}396}$$

$\therefore$ $$x_c(18{,}396) = 200 \times 150 \times 100 - \frac{1}{2} \times 100 \times 75 \times 166.67 - \frac{\pi}{4} \times 100^2 x_c$$

$$x_c = \frac{23{,}75000}{26{,}250} = \mathbf{90.5 \text{ mm}}$$ **Ans.**

Similarly,

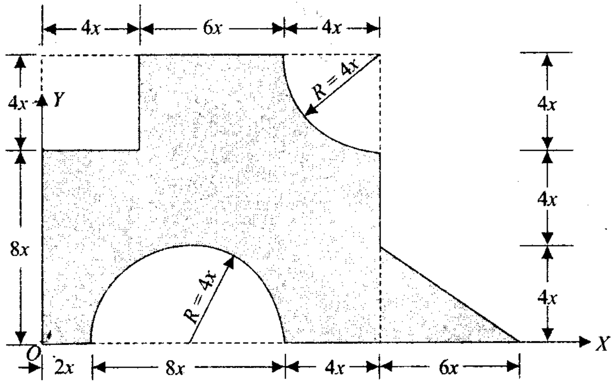
$$18{,}396 y_c = 200 \times 150 \times 75 - \frac{1}{2} \times 100 \times 75 \times (150 - 25) - \frac{\pi}{4} \times 100^2 y_c$$

$\therefore$ $$y_c = \frac{17{,}81250.0}{26{,}250} = \mathbf{67.9 \text{ mm}}$$ **Ans.**

Centre of the circle should be located at (90.48, 67.86) so that this point will be the centroid of the remaining shaded area shown in Fig. 7.16.

Note: The centroid of the given figure will coincide with the centroid of the figure without circular hole. Hence, the centroid of the given figure may be obtained by determining the centroid of the figure without the circular hole, also.

Example 7.8 Determine the coordinates of the centroid of the plane area shown in Fig. 7.17 with reference to the axes shown. Take $x = 40$ mm.



The composite figure is divided into the following simple figures:

(1) A rectangle

$$A_1 = (14x) \times (12x) = 168x^2$$

$$x_1 = 7x;\ y_1 = 6x$$

(2) A triangle

$$A_2 = \frac{1}{2}(6x) \times (4x) = 12x^2$$

$$x_2 = 14x + 2x = 16x$$

$$y_2 = \frac{4x}{3}$$

(3) A rectangle to be subtracted

$$A_3 = (-4x) \times (4x) = -16x^2$$

$$x_3 = 2x;\ y_3 = 8x + 2x = 10x$$

(4) A semicircle to be subtracted

$$A_4 = -\frac{1}{2}\pi\,(4x)^2 = -8\pi x^2$$

$$x_4 = 6x$$

$$y_4 = \frac{4R}{3\pi} = 4 \times \frac{(4x)}{3\pi} = \frac{16x}{3\pi}$$

(5) A quarter of a circle to be subtracted

$$A_5 = -\frac{1}{4} \times \pi(4x)^2 = -4\pi x^2$$

$$x_5 = 14x - \frac{4R}{3\pi} = 14x - (4)\left(\frac{4x}{3\pi}\right) = 12.3023x$$

$$y_5 = 12x - 4 \times \left(\frac{4x}{3\pi}\right) = 10.3023x$$

Total area $A = 168x^2 + 12x^2 - 16x^2 - 8\pi x^2 = 4\pi x^2$

$= 126.3009\ x^2$

$$\bar{x} = \frac{\sum A_i x_i}{A}$$

$$\Sigma A_i x_i = 168x^2 \times 7x + 12x^2 \times 16x - 16x^2 \times 2x - 8\pi x^2 \times 6x - 4\pi x^2 \times 12.3023x = 1030.6083x^3$$

$$\therefore \quad \bar{x} = \frac{1030.6083x^3}{126.3009x^2}$$

$= 8.1599x = 8.1599 \times 40$ (since $x = 40$ mm)

$= 326.40$ mm

$$\bar{y} = \frac{\sum A_i y_i}{A}$$

$$\Sigma A_i x_i = 168x^2 \times 6x + 12x^2 \times \frac{4x}{3} - 16x^2 \times 10x - 8\pi x^2 \times \frac{16x}{3\pi} - 4\pi x^2 \times 10.3023x$$

$= 691.8708x^3$

$$\therefore \quad \bar{y} = \frac{691.8708}{126.3009x^2}$$

$= 5.4780\ x$

$= 219.12$ mm (since $x = 40$ mm)

Centroid is at **(326.40, 219.12)** **Ans.**

mportant Definitions and Concepts

1. Centre of gravity is the point through which the resultant of total weight acts for any orientation of the body.
2. Centroid is the point in the plane area such that for any axis through that point moment of area is zero.

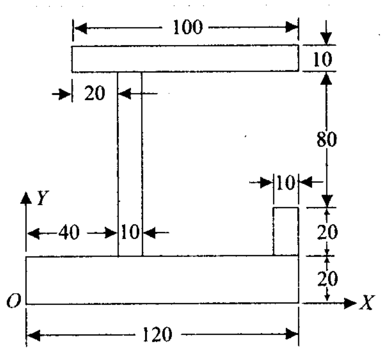
Important Formulae

$$\bar{y} = \frac{\sum A_i y_i}{A} = \frac{\int y\,dA}{A} \text{ and } \bar{x} = \frac{\sum A_i x_i}{A} = \frac{\int x\,dA}{A}$$

roblems for Exercise

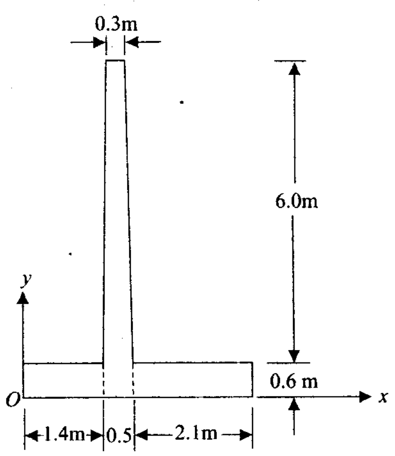
7.1 Determine the centroid of the built-up section in Fig. 7.18. Express the coordinates of centroid with respect to x and y axes shown.

[**Ans.** $\bar{x}$ = 48.9 mm; $\bar{y}$ = 61.3 mm]



7.2 Determine the centroid of the reinforced concrete retaining wall section shown in Fig. 7.19.

[**Ans.** $\bar{x}$ = 1.848 m; $\bar{y}$ = 1.825 m]



7.3 Find the coordinates of the centroid of the shaded area with respect to the axes shown in Fig. 7.20.

[**Ans.** x = 44 mm; y = 70.1 mm]

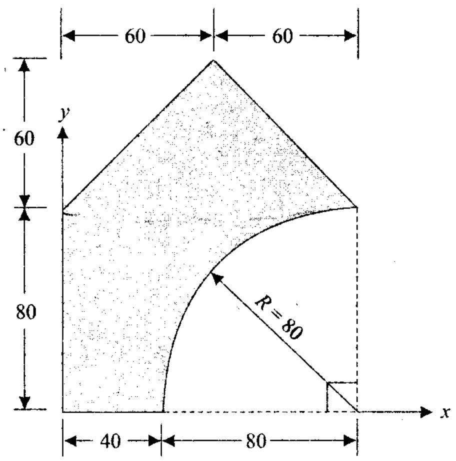


Fig. 7.20

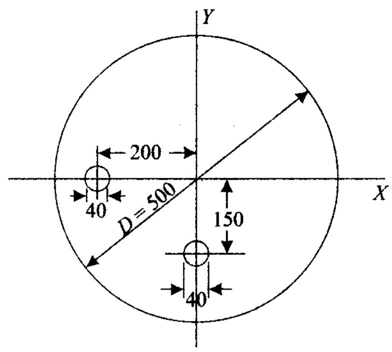


Fig. 7.21

7.4 A circular plate of uniform thickness and of diameter 500 mm as shown in Fig. 7.21 has two circular holes of 40 mm diameter each. Where should a 80 mm diameter hole be drilled so that the centre of gravity of the plate will be at the geometric centre?

[**Ans.** x = 50 mm; y = 37.5 mm]

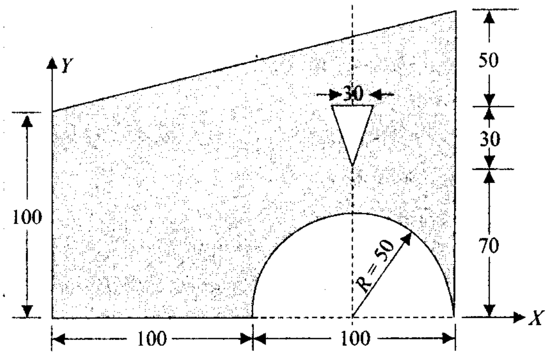


Fig. 7.22

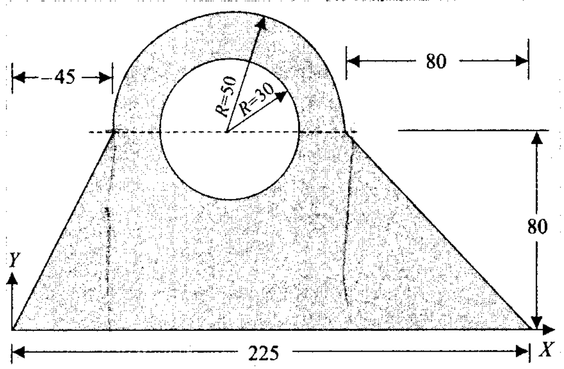


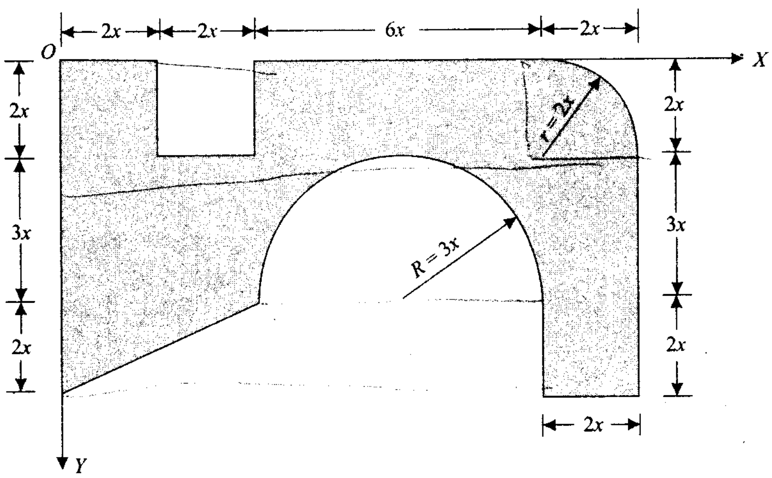
Fig. 7.23

7.5 With respect to the coordinate axes x and y locate the centroid of the shaded area shown in Fig. 7.22.

[**Ans.** $\bar{x}$ = 97.47 mm; $\bar{y}$ = 70.77 mm]

7.6 Locate the centroid of the plane area shown in Fig. 7. 23.

[**Ans.** $\bar{x}$ = 104.10 mm; $\bar{y}$ = 44.30 mm]



7.7 Determine the coordinates of the centroid of shaded area shown in Fig. 7.24 with respect to the corner point O. Take $x = 40$ mm.

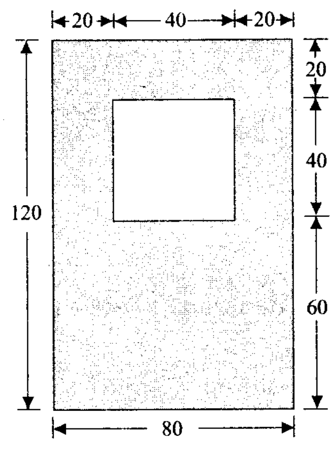
[Ans. $\bar{x} = 260.07$ mm; $\bar{y} = 113.95$ mm]

7.8 $ABCD$ is a square section of sides 100 mm. Determine the ratio of moment of inertia of the section about centroidal axis parallel to a side to that about diagonal AC.

[Ans. 1]

7.9 The cross-section of a rectangular hollow beam is as shown in Fig. 7.53. Determine the polar moment of inertia of the section about centroidal axes.

[Ans. I_{xx} = 1,05,38667 mm^4; I_{yy} = 49,06667 mm^4; I_{zz} = 1,54,45334 mm^4]



7.10 The cross-section of a prestressed concrete beam is shown in Fig. 7.26. Calculate the moment of inertia of this section about the centroidal axes parallel to and perpendicular to top edge. Also determine the radii of gyration.

[**Ans.** I_{xx} = 1.15668 × 10^{10} mm^4; k_{xx} = 231.95 mm; I_{yy} = 8.75729 × 10^9 mm^4; k_{yy} = 201.82 mm]

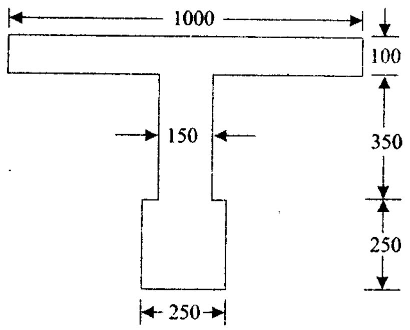


Fig. 7.26

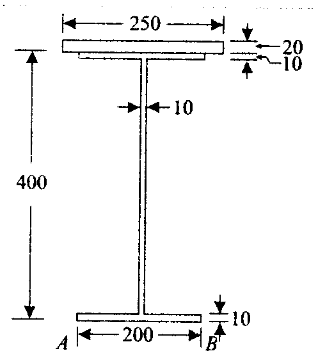


Fig. 7.27

7.11 The strength of a 400 mm deep and 200 mm wide *I*-beam of uniform thickness 10 mm, is increased by welding a 250 mm wide and 20 mm thick plate to its upper flange as shown in Fig. 7.27. Determine the moment of inertia and the radii of gyration of the composite section with respect to centroidal axes parallel to and perpendicular to the bottom edge *AB*.

[**Ans.** I_{xx} = 3.32393 × 10^8 mm^4; k_{xx} = 161.15 mm; I_{yy} = 3,94,06667 mm^4; k_{yy} = 55.49 mm]

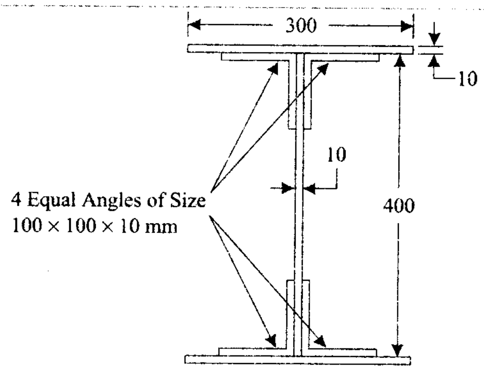


Fig. 7.28

7.12 The cross-section of a gantry girder is as shown in Fig. 7.28. It is made up of an *I*-section of depth 450 mm, flange width 200 mm and a channel of size 400 mm × 150 mm. Thickness of all members is 10 mm. Find the moment of inertia of the section about the horizontal centroid axis.

[**Ans.** I_{xx} = 4.2198 × 10^8 mm^4]

7.13 A plate girder is made up of a web plate of size 400 mm × 10 mm, four angles of size 100 mm × 100 mm × 10 mm and cover plates of size 300 mm × 10 mm as shown in Fig. 7.57. Determine the moment of inertia about horizontal and vertical centroidal axes.

[**Ans.** I_{xx} = 5.35786 × 10^8 mm^4; I_{yy} = 6,08,50,667 mm^4]

CHAPTER 8

Equilibrium of Coplanar Concurrent Force System

A body is said to be in equilibrium when its static position or its motion with uniform velocity is not altered by the system of forces acting on it. The bodies are subjected to applied forces, self weight and reactions from the adjoining bodies. The analysis of equilibrium conditions of bodies is to determine the reaction from the adjoining bodies in contact. In this chapter it is aimed to analyse the bodies in equilibrium under coplanar concurrent force system. Since it is concurrent force system case, the size of the body will not come into analysis. In other words, the body is treated as a particle. This chapter may be titled as Equilibrium of Particles also. In this chapter after introducing to this topic a number of problems are solved.

8.1 TYPES OF FORCES ACTING ON A BODY

Various forces acting on a body may be grouped as

- Applied forces and
- Non-applied forces

8.1.1 Applied Forces

Applied forces are the forces applied externally to a body. Each force has got a point of contact with the body. If a person stands on a ladder, his weight is an applied force to the ladder. If a temple car is pulled, the force in the rope is an applied force for the car.

8.1.2 Non-applied Forces

There are two types of nonapplied forces, namely

- Self weight and
- Reactions.

(i) *Self weight:* Every body is subjected to gravitational attraction and hence has got self weight, given by the expression

$$W = mg \qquad \text{Eqn. (8.1)}$$

where m is the mass of the body and g is the gravitational attraction (9.81 m/sec^2 near the earth surface). Self weight always acts in vertically downward direction. This force is treated as a vertically downward force acting through the centre of gravity of the body. If self weight is very small compared to other forces, it may be neglected.

(ii) *Reactions:* Reactions are the self adjusting forces developed by other bodies which are in contact with the body under consideration. According to Newton's third law of motion, reactions are equal and opposite to actions. The reactions adjust themselves to keep the body in equilibrium.

If the surface of contacts are smooth, the direction of reaction is normal to the surface of contact. If the surface of contact is rough, apart from normal reaction, there will be frictional reaction in tangential direction. Hence the resultant reactions will not be in the normal direction. However, in this chapter all surfaces of contact are treated as smooth, since the topic on friction is going to be introduced only in Chapter 10.

8.2 FREE BODY DIAGRAM

For the analysis of equilibrium condition it is necessary to isolate the body under consideration from the other bodies in contact and draw all forces acting on the body. For this, first the body is drawn and then all applied forces, self weight and the reactions from the other bodies in contact are drawn. This type of diagram of the body in which the body under consideration is freed from all contact surfaces and is shown with all the forces on it (including self weight, reactions and applied forces) is called Free Body Diagram (FBD). Free body diagrams are shown for few typical cases in Table 8.1.

8.3 EQUATIONS OF EQUILIBRIUM

According to Newton's first law a body is in static equilibrium or in motion with uniform velocity unless resultant forces act on it. Hence, if a body is in equilibrium we can conclude that the resultant for various fources in the system is zero. In case of coplanar concurrent force system it means

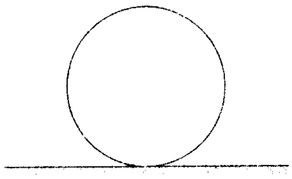
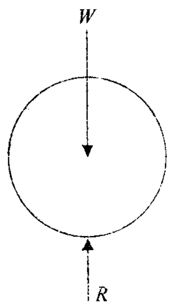
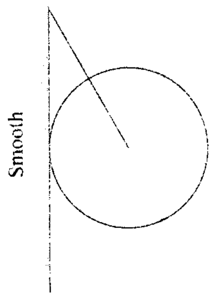
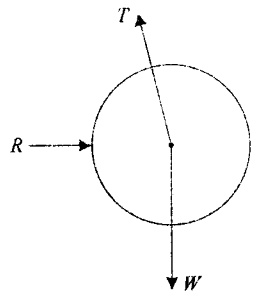
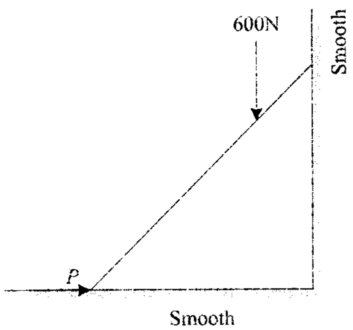
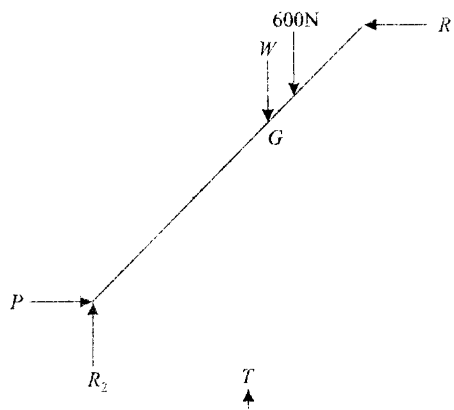
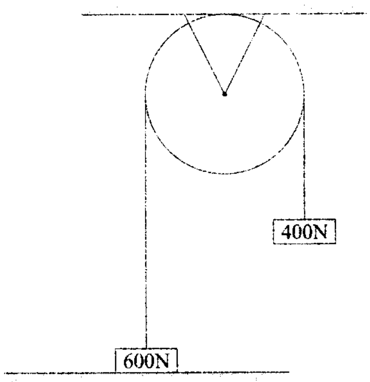
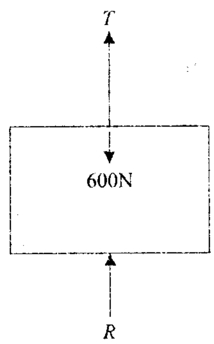
$$\sum F_x = 0$$

and

$$\sum F_y = 0 \qquad \text{Eqn. (8.2)}$$

It may be observed that $\Sigma F_x = 0$ means, $R \cos \alpha = 0$. In other words the resultant is not having any component in x-direction. This condition ensures that there is no resultant force in any direction, except y-direction in which $\alpha = 90°$. Hence the condition $\Sigma F_y = 0$ also should be satisfied to ensure that the resultant R does not exist in any direction. Therefore, if a body is in equilibrium under the action of coplanar concurrent system, equation 8.2 should be satisfied.

Table 8.1 Free Body Diagrams [illegible]

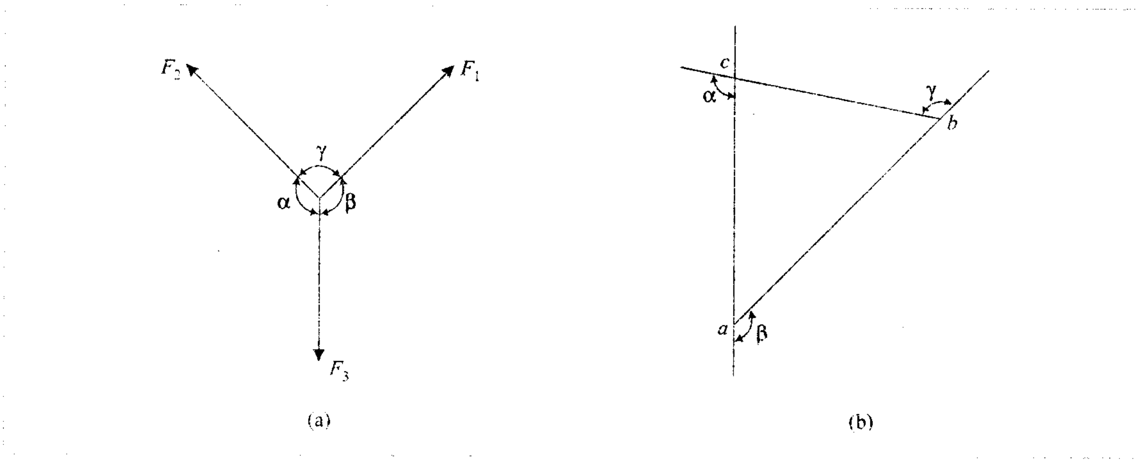
Reacting bodies	*FBD required for*	*FBD*
	Ball	
	Ball	
	Ladder	
	Block weighing 600 N	

8.4 LAMI'S THEOREM

If a body is in equilibrium, under the action of coplanar concurrent forces, it may be analysed using equations of equilibrium (eqn. 8.2). However, if the body is in equilibrium under the action of only three forces, Lami's theorem can be used conveniently.

Lami's theorem states that *if a body is in equilibrium under the action of only three forces, each force is proportional to the sine of the angle between the other two forces.* Thus for the system of forces shown in Fig. 8.1(a),

$$\frac{F_1}{\sin\alpha} = \frac{F_2}{\sin\beta} = \frac{F_3}{\sin\gamma} \qquad \text{Eqn. (8.3)}$$



Proof. Draw the three forces F_1, F_2 and F_3 one after the other in direction and magnitude starting from point '*a*'. Since the body is in equilibrium the resultant should be zero, which means the last point of force diagram should coincide with '*a*'. Thus it results in a triangle of forces *abc* as shown in Fig. 8.1(b). Now the external angles at *a*, *b* and *c* are equal to β, γ and α, since *ab*, *bc* and *ca* are parallel to F_1, F_2 and F_3 respectively. In the triangle of forces *abc*,

$$ab = F_1$$

$$bc = F_2$$

and

$$ca = F_3$$

Applying sine rule for the triangle *abc*,

$$\frac{ab}{\sin(180-\alpha)} = \frac{bc}{\sin(180-\beta)} = \frac{ca}{\sin(180-\gamma)}$$

i.e.,

$$\frac{F_1}{\sin\alpha} = \frac{F_2}{\sin\beta} = \frac{F_3}{\sin\gamma}$$

Note: When a body is in equilibrium under the action of only three forces, those three forces must be concurrent. Proof of this statement is given in the next chapter. However, this concept has been used in this article also.

Example 8.1 A sphere weighing 100 N is tied to a smooth wall by a string as shown in Fig. 8.2(a). Find the tension T in the string and the reaction R from the wall.

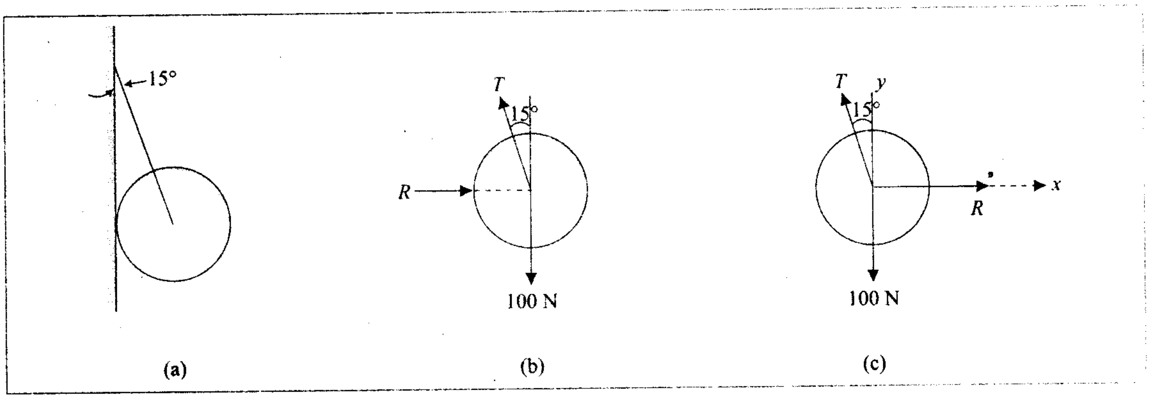


Fig. 8.2

Solution. Free body diagram of the sphere is as shown in Fig. 8.2(b). Figure 8.2(c) shows all the forces acting away from the centre of the ball, which is permissible as per the law of transmissibility of forces. Applying Lami's theorem to the system of forces, we get

$$\frac{T}{\sin 90} = \frac{R}{\sin(180-15)} = \frac{100}{\sin(90+15)}$$

$$T = \mathbf{103.5\ N} \qquad \textbf{Ans.}$$

and $$R = \mathbf{26.8\ N} \qquad \textbf{Ans.}$$

The above problem may be solved by using equations of equilibrium also. Taking horizontal direction as x-axis and vertical direction as y-axis as shown in Fig. 8.2(c), we get

$$\sum F_y = 0 \rightarrow$$

$$T\cos 15 - 100 = 0$$

$\therefore$ $$T = \mathbf{103.5\ N} \qquad \textbf{Ans.}$$

$$\sum F_x = 0 \rightarrow$$

$$R - T\sin 15 = 0$$

$\therefore$ $$R = 103.5 \sin 15 = \mathbf{26.8\ N} \qquad \textbf{Ans.}$$

Note the following points:

1. The starting can have only tension in it (it can pull a body), but cannot have compression in it (cannot push a body).
2. The wall reaction is a push, but not a pull on the body.

3. The line of action of reactions should be determined accurately, but the direction can be assumed. If assumed direction is correct the value comes out to be positive. If that is exactly opposite, the value comes out to be negative. Hence in such cases, the result may be reported with the reversed direction.

Example 8.2 Determine the horizontal force F to be applied to the block weighing 1500 N to hold it in position on a smooth inclined plane AB which makes an angle 30° with the horizontal [Ref. Fig. 8.3(a)].

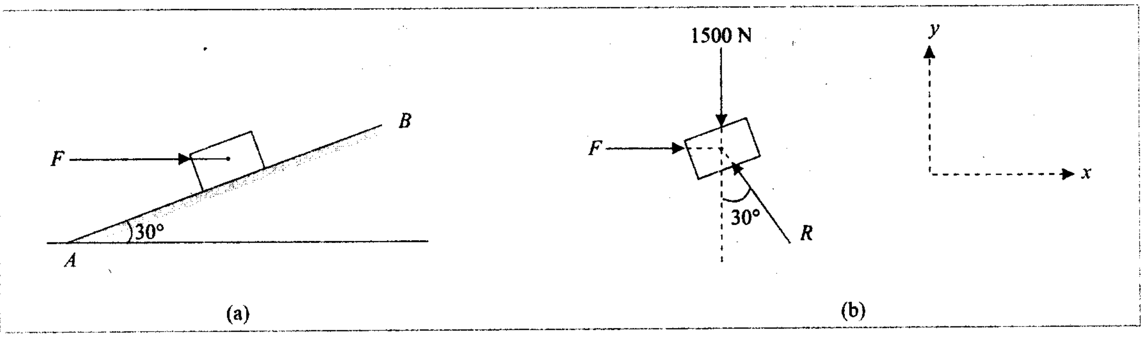


Fig. 8.3

Solution. The body is in equilibrium under the action of applied force F, self weight 1500 N and reaction R from the plane. Applied force is horizontal and self weight is vertically downward. Reaction is normal to the plane AB, since the plane AB is smooth. Since plane makes 30° to horizontal, normal to it makes 60° to horizontal i.e., 30° to vertical [Ref. Fig. 8.3(b)].

$$\sum F_y = 0 \rightarrow$$

$$R \cos 30 - 1500 = 0$$

$$\therefore \quad R = \frac{1500}{\cos 30} = 1732.0 \text{ N}$$ **Ans.**

$$\sum F_x = 0 \rightarrow$$

$$F - R \sin 30 = 0$$

$$\therefore \quad F = R \sin 30 = 1732 \sin 30$$

i.e., $\mathbf{F = 866\ N}$ **Ans.**

Note: Since the body is in equilibrium under the action of only three forces, the above problem can be solved using Lami's theorem also, as shown below:

$$\frac{R}{\sin 90} = \frac{F}{\sin (180 - 30)} = \frac{1500}{\sin (90 + 30)}$$

$\therefore$ $\mathbf{R = 1732\ N}$ **and** $\mathbf{F = 866\ N}$ **Ans.**

Example 8.3 Find the forces developed in the wires, supporting an electric fixture as shown in Fig. 8.4(a).

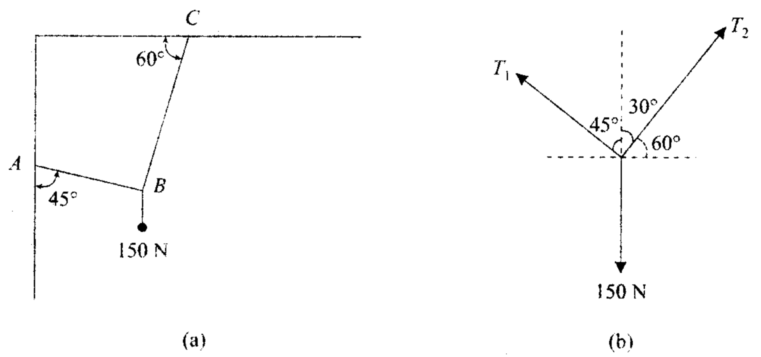


Fig. 8.4

Solution. Let the forces developed in the wires BA and BC be T_1 and T_2 as shown in Fig. 8.4(b). Applying Lami's theorem to the system of forces, we get

$$\frac{T_1}{\sin(90+60)} = \frac{T_2}{\sin(180-45)} = \frac{150}{\sin(45+30)}$$

$\therefore$ $\quad$ $\mathbf{T_1 = 77.6\ N}$ **and** $\mathbf{T_2 = 109.8\ N}$ **Ans.**

Example. 8.4 A 200 N sphere is resting in a trough as shown in Fig. 8.5(a). Determine the reactions developed at contact surfaces. Assume all contact surfaces are smooth.

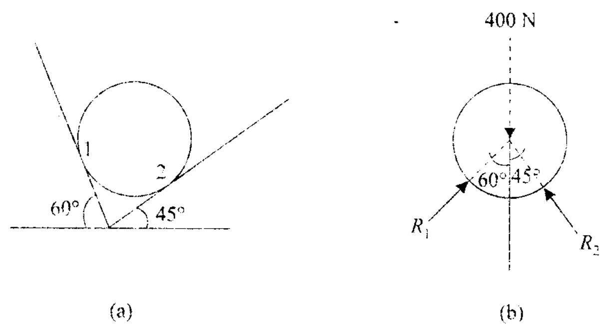


Fig. 8.5

Solution. At contact point 1, the surface of contact is making 60° to horizontal. Hence the reaction R_1 which is normal to it makes 60° with the vertical. Similarly the reaction R_2 at contact point 2 makes 45° to the vertical. FBD is as shown in Fig. 8.5(b).

Applying Lami's theorem to the system of forces, we get

$$\frac{R_1}{\sin(180-45)} = \frac{R_2}{\sin(180-60)} = \frac{400}{\sin(60+45)}$$

$\therefore$ $R_1 = \mathbf{292.8\ N}$ and $R_2 = \mathbf{358.6\ N}$ **Ans.**

Example. 8.5 A roller weighing 10 kN rests on a smooth horizontal floor and is connected to the floor by the bar *AC* as shown in Fig. 8.6(a). Determine the force in the bar *AC* and reaction from the floor, if the roller is subjected to a horizontal force of 5 kN and an inclined force of 7 kN as shown in Fig. 8.6.

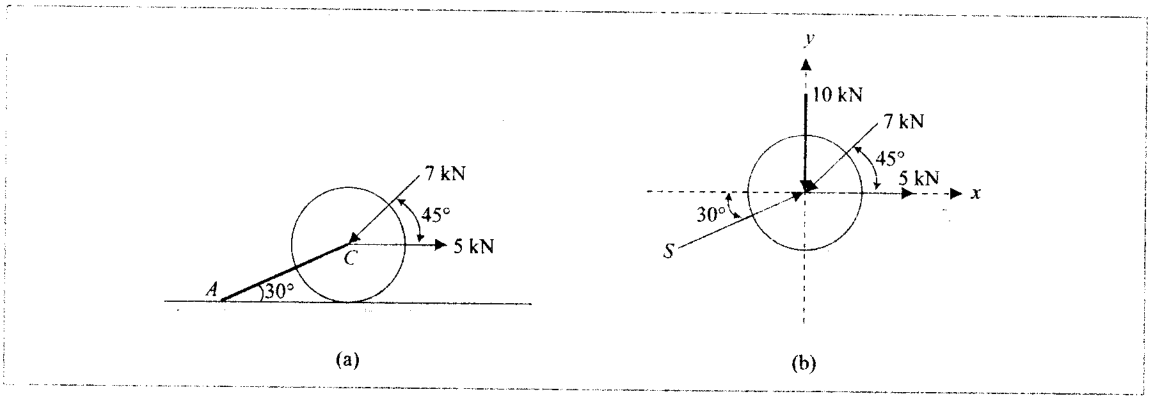


Fig. 8.6

Solution. A bar can develop tensile force or compressive force. Let the force developed be a compressive force *S* (push on the cylinder). Free body diagram of the cylinder is as shown in Fig. 8.6(b).

Since there are more than three forces in the system, Lami's equations cannot be used. Consider the equilibrium equations.

$$\sum F_H = 0 \rightarrow$$

$$S\cos 30 + 5 - 7\cos 45 = 0$$

$$S = \frac{7\cos 45 - 5}{\cos 30} = -0.058 \text{ kN}$$

Since the value is negative the reaction from the bar is not push, but it is a pull (tensile force in the bar) of magnitude **0.058 kN**. **Ans.**

Referring to Fig. 8.6(b),

$$\sum F_y = 0 \rightarrow$$

$$R - 10 - 7\sin 45 + S\sin 30 = 0$$

$$R = 10 + 7\sin 45 - S\sin 30$$

$$= 10 + 7 \sin 45 - (-0.058) \sin 30$$

i.e., $R = 14.980$ kN **Ans.**

Example 8.6 A roller of radius $r = 300$ mm and weighing 2000 N is to be pulled over a curb of height 150 mm, as shown in Fig. 8.7(a) by applying a horizontal force F applied to the end of a string wound around the circumference of the roller. Find the magnitude of force F required to start the roller move over the curb. What is the least pull F through the centre of the wheel to just turn the roller over the curb?

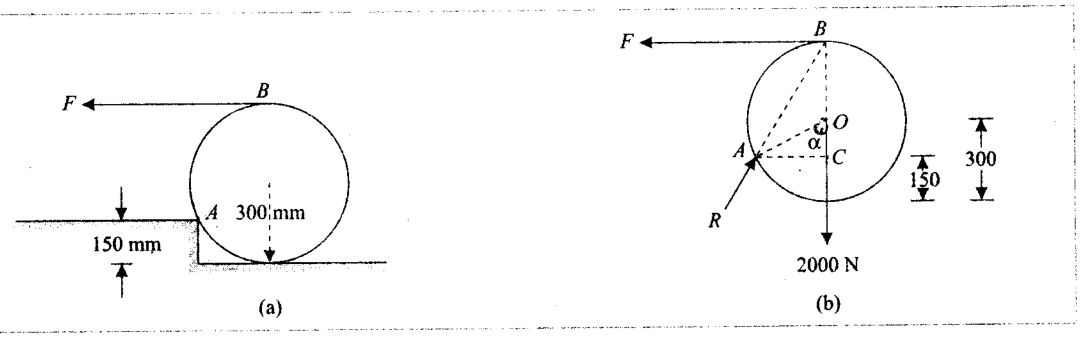


Fig. 8.7

Solution. When the roller is about to turn over the curb, the contact with the floor is lost and hence, there is no reaction from the floor. The body is in equilibrium under the action of three forces, namely,

(i) Applied force F, which is horizontal
(ii) Self weight, which is vertically downward, acting through the centre of roller and
(iii) Reaction R from the edge of the curb. Since the body is in equilibrium under the action of only three forces, they must be concurrent. It means the reaction at edge A of curb passes through the point B as shown in the Fig. 8.7(b).

Referring to Fig. 8.7(b) in which O is the centre of roller and AC horizontal line,

$$\cos \alpha = \frac{OC}{AO} = \frac{300 - 150}{300} = \frac{1}{2}$$

$$\therefore \quad \alpha = 60°$$

Now, in ΔAOB,

$$\angle OAB = \angle OBA \text{ since } OA = OB = \text{radius of roller.}$$

but $\angle OAB + \angle OBA = \alpha$

$\therefore$ $2\angle OBA = 60°$

or $\angle OBA = 30°$

i.e., reaction makes 30° with the vertical.

$$\sum V = 0 \rightarrow$$

$$R \cos 30 - 2000 = 0$$

$\therefore$ $$R = 2309.4 \text{ N}$$

$$\sum H = 0 \rightarrow$$

$$F - R \sin 30 = 0$$

or $$F = 2309.4 \sin 30 = \mathbf{1154.7\ N}$$ **Ans.**

Least force through the centre of roller:

In this case the reaction from the curb must pass through the centre of the roller since the other two forces pass through that point. Its inclination to vertical is $\theta = 60°$.

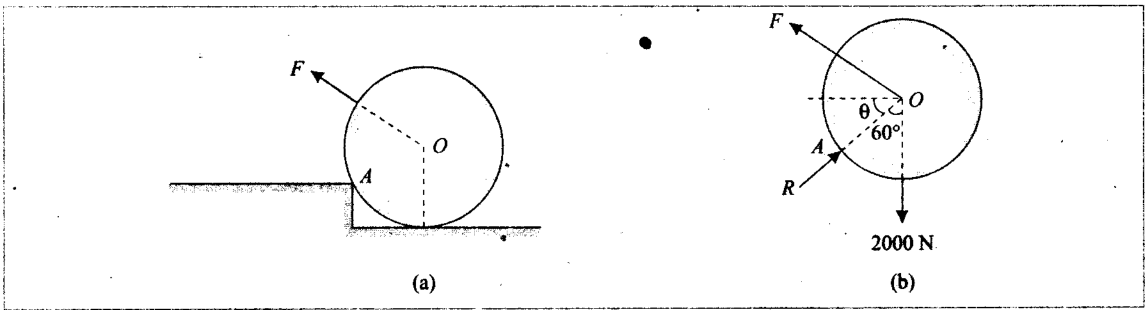


Fig. 8.8

Solution. Let force F make angle θ with the horizontal as shown in Fig. 8.8(b).

$$\sum F_H = 0 \rightarrow F \cos \theta = R \sin 60$$

$$\sum F_V = 0 \rightarrow R \sin \theta + R \cos 60 - W = 0$$

i.e., $$F \sin \theta + \frac{F \cos \theta}{\sin 60} \times \cos 60 = W$$

i.e., $$F[\sin \theta + \cot 60 \cos \theta] = W$$

$\therefore$ $$\sin \theta + \cot 60 \cos \theta = \frac{W}{P}$$

For $\frac{W}{P}$ to be maximum i.e. P to be least,

$$\frac{d}{d\theta}\left[\frac{W}{P}\right] = 0$$

i.e., $$\cos \theta + \cot 60 (- \sin \theta) = 0$$

$$\cos \theta = \cot 60 \sin \theta$$

or $\cot \theta = \cot 60$

i.e., $\theta = 60°$

i.e., F is least when it is at right angles to the reaction R.

$$\therefore \quad P_{min} = \frac{W}{\sin 60 + \cot 60° \cos 60} = \frac{2000 \sin 60}{\sin^2 60 + \cos^2 60}$$

$$= \mathbf{1732\ N}$$ **Ans.**

Example 8.7 The frictionless pulley A shown in Fig. 8.9(a), is supported by two bars AB and AC which are hinged at B and C to a vertical wall. The flexible cable hinged at D goes over the pulley and supports a load at 20 kN at G. The angle made by various members of the system are as shown in Fig. 8.9. Determine the forces in the bars AB and AC. Neglect the size of the pulley.

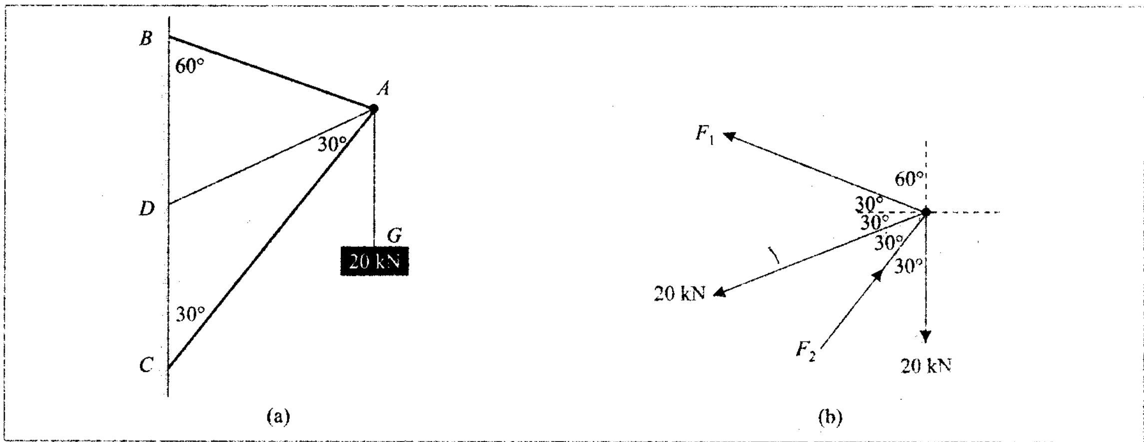


Fig. 8.9

Solution. Since pulley is frictionless, same force exist throughout the flexible cable. Hence the force in AD is also 20 kN as shown in Fig. 8.9(b). From the figure it may be observed that AC is perpendicular to AB. Selecting AB and AC as Cartesian x and y axes,

$$\sum F_x = 0 \rightarrow$$

$$F_1 + 20 \sin 30 - 20 \sin 30 = 0$$

$$\therefore \quad \mathbf{F_1 = 0}$$ **Ans.**

$$\sum F_y = 0 \rightarrow$$

$$- F_2 + 20 \cos 30 + 20 \cos 30 = 0.$$

$$\therefore \quad F_2 = 40 \cos 30 = \mathbf{34.6\ N}$$ **Ans.**

8.5 EQUILIBRIUM OF CONNECTED BODIES

When two or more bodies are in contact with one another, the system of forces appear as though it is a nonconcurrent forces system. However, when each body is considered separately, in many situations it turns out to be a set of concurrent force system. In such cases, first the body subjected to only two unknown forces is analysed and then it is followed by the analysis of other connected bodies. This type of examples are illustrated below.

Example 8.8 A system of connected flexible cables shown in Fig. 8.10(a) is supporting two vertical forces 200 N and 250 N at points *B* and *D*. Determine the forces in various segments of the cable.

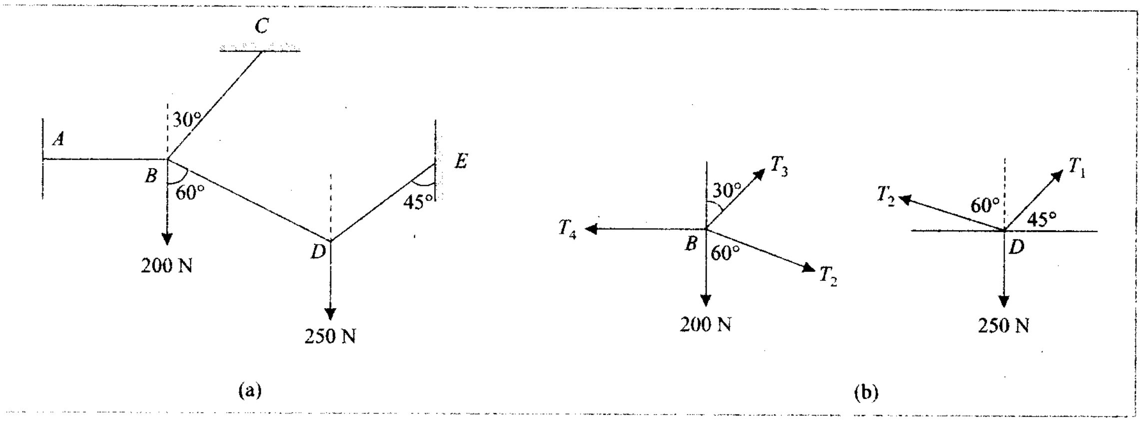


Fig. 8.10

Soluion. Free body diagrams of points *B* and *D* are as shown in Fig. 8.10(b). Le the forces in the members be as shown in the figure.

Applying Lami's theorem to the system of forces at point *D*, we get

$$\frac{T_1}{\sin(180-60)} = \frac{T_2}{\sin(90+45)} = \frac{250}{\sin(60+45)}$$

∴ $\quad T_1 = 224.1 \text{ N}$ and $T_2 = 183 \text{ N}$ **Ans.**

Now, consider the system of forces acting at *B*.

$$\sum F_V = 0 \rightarrow$$

$$T_3 \cos 30 - T_2 \cos 60 - 200 = 0$$

$$T_3 \cos 30 = T_2 \cos 60 + 200 = 183 \cos 60 + 200 = 291.6$$

∴ $\quad T_3 = 336.6 \text{ N}$ **Ans.**

$$\sum F_H = 0 \rightarrow$$

$$-T_4 + T_3 \sin 30 + T_2 \sin 60 = 0$$

∴ $\quad T_4 = 336.6 \sin 30 + 183 \sin 60 = \mathbf{326.8 \text{ N}}$ **Ans.**

Example 8.9 Rope AB shown in Fig. 8.11(b) is 4.5 m long and is connected at two points A and B at the same level 4 m apart. A load of 1500 N is suspended from a point C on the rope at 1.5 m from A. What load connected at point D on the rope, 1 m from B will be necessary to keep the position CD level?

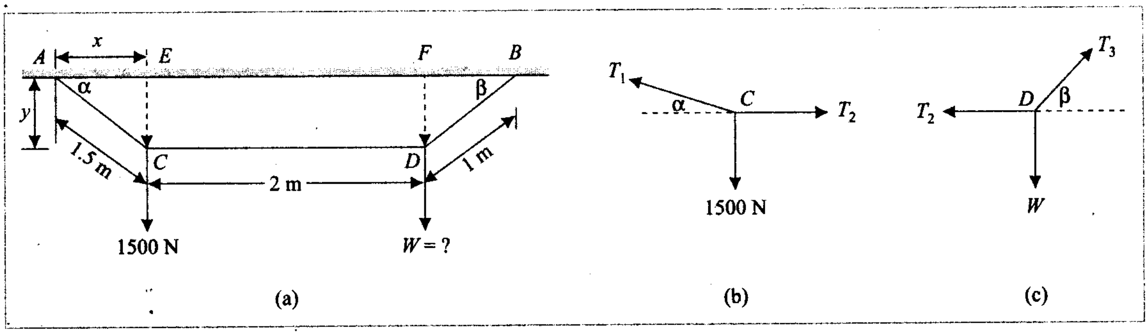


Fig. 8.11

Solution. Drop perpendiculars CE and DE on AB.

Let $CE = y$ and $AE = x$

From ΔAEC,

$$x^2 + y^2 = 1.5^2 = 2.25 \qquad \text{...(i)}$$

Now, $AB = 4$ m

and $AC + CD + DB = 4.5$ m

$\therefore$ $CD = 4.5 - 1.5 - 1.0$, since $AC = 1.5$ m and $DB = 1$ m

$= 2.0$ m

$\therefore$ $EF = 2.0$ m

Now, $BF = AB - (AE + EF)$

$= 4 - [x + 2] = 2 - x \qquad \text{...(ii)}$

From ΔBFD,

$$BF^2 + DF^2 = 1^2$$

$$(2 - x)^2 + y^2 = 1 \qquad \text{...(iii)}$$

Subtracting eqn. (iii) from eqn. (i), we get

$$x^2 - (2 - x)^2 = 1.25$$

i.e., $x^2 - 4 + 4x - x^2 = 1.25$

$$x = 1.3125 \text{ m}$$

$\therefore$ $\alpha = \cos^{-1} \dfrac{1.3125}{1.5} = 28.955°$

and $\beta = \cos^{-1} \dfrac{2 - 1.3125}{1} = 46.567°$

Applying Lami's theorem to the system of forces acting at point C [Ref. Fig. 8.11(b)],

$$\frac{T_1}{\sin 90} = \frac{T_2}{\sin(90+28.955)} = \frac{1500}{\sin(180-28.955)}$$

$\therefore$ $T_1 = 3098.4$ N

$T_2 = 2711.1$ N

Now, applying Lami's theorem to the system of forces at B [Fig. 8.11(c)],

$$\frac{T_3}{\sin 90} = \frac{W}{\sin(180-46.567)} = \frac{T_2}{\sin(90+46.567)}$$

$\therefore$ $T_3 = 3943.4$ N

and **$W = 2863.6$ N** **Ans.**

Example. 8.10 A wire rope is fixed at two points A and D as shown in Fig. 8.12(a). Weights 20 kN and 30 kN are attached to it at B and C, respectively. The weights rest with portions AB and BC inclined at 30° and 50°, respectively, to the vertical as shown in the figure. Find the tension in segments AB, BC and CD at the wire. Determine the inclination of the segment CD to vertical.

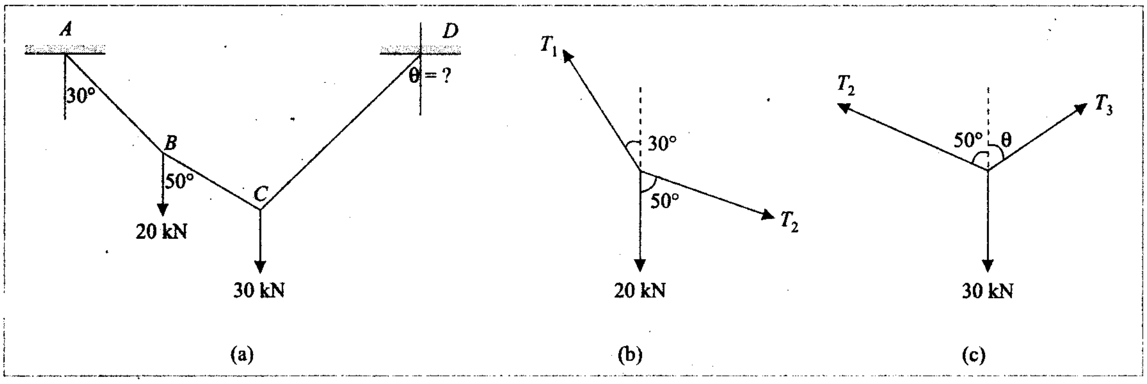


Fig. 8.12

Solution. Figures 8.12(b) and 8.12(c) show free body diagrams of points B and C. Applying Lami's theorem for the system of forces B,

$$\frac{T_1}{\sin 50} = \frac{T_2}{\sin(180-30)} = \frac{20}{\sin(180+30-50)}$$

$\therefore$ **$T_1 = 44.80$ kN and $T_2 = 29.2$ kN** **Ans.**

Consider the equilibrium of forces at C.

$$\sum F_H = 0 \rightarrow$$

$$T_3 \sin\theta = T_2 \sin 50$$

$$= 22.4 \qquad \text{...(i)}$$

$$\sum F_V = 0 \rightarrow$$

$T_3 \cos\theta + T_2 \cos 50 - 30 = 0$

i.e., $T_3 \cos\theta = 30 - 29.2 \cos 50$

$= 11.20$ ...(ii)

From eqns. (i) and (ii), we get

$\tan\theta = 2$

$\therefore$ $\theta = \mathbf{63.43°}$ **Ans.**

Substituting this value in eqn. (i), we get

$T_3 = \mathbf{25.04\ kN}$ **Ans.**

Example 8.11 A wire is fixed at *A* and *D* as shown in Fig. 8.13(a). Weights 20 kN and 25 kN are supported at *B* and *C* respectively. When equilibrium is reached it is found that inclination of *AB* is 30° and that of *CD* is 60° to the vertical. Determine the tension in the segments *AB*, *BC* and *CD* of the rope and also the inclination of *BC* to the vertical.

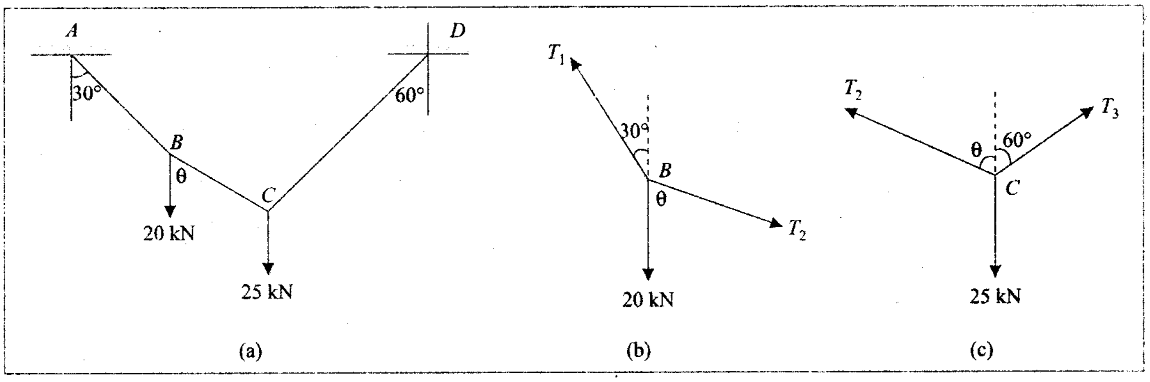


Fig. 8.13

Solution. Free body diagrams of points *B* and *C* are shown in Figs. 8.13(b) and (c) respectively. Considering equilibrium of point *B*, we get

$$\sum F_H = 0 \rightarrow \qquad T_2 \sin\theta - T_1 \sin 30 = 0$$

i.e., $T_2 \sin\theta = T_1 \sin 30$ ...(i)

$$\sum F_V = 0 \rightarrow -T_2 \cos\theta + T_1 \cos 30 - 20 = 0$$

i.e., $T_2 \cos\theta = T_1 \cos 30 - 20$ ...(ii)

Considering the equilibrium of point *C*,

$$\sum F_H = 0 \rightarrow \qquad T_3 \sin 60 - T_2 \sin\theta = 0$$

i.e., $T_2 \sin\theta = T_3 \sin 60.$ ...(iii)

$$\sum F_V = 0 \rightarrow \qquad T_3 \cos 60 + T_2 \cos\theta - 25 = 0$$

i.e., $$T_2 \cos\theta = -T_3 \cos 60 + 25 \qquad \text{...(iv)}$$

From eqns. (i) and (iii), we get

$$T_1 \sin 30 = T_3 \sin 60°$$

i.e., $$T_1 = \sqrt{3}\, T_3 \qquad \text{...(v)}$$

From eqns. (ii) and (iv), we get,

$$T_1 \cos 30 - 20 = 25 - T_3 \cos 60$$

$$\sqrt{3}\, T_3 \frac{\sqrt{3}}{2} - 20 = 25 - T_3 \times 0.5$$

i.e., $$2T_3 = 45$$

or $$\mathbf{T_3 = 22.5\ kN} \qquad \textbf{Ans.}$$

From eqn. (v) $$\mathbf{T_1 = 38.97\ kN} \qquad \textbf{Ans.}$$

From eqn. (i) $$T_2 \sin\theta = 19.48$$

and from eqn. (ii) $$T_2 \cos\theta = 13.75$$

$\therefore$ $$\tan\theta = 1.416$$

$\therefore$ $$\mathbf{\theta = 54.78°} \qquad \textbf{Ans.}$$

and $$\mathbf{T_2 = 23.84\ kN} \qquad \textbf{Ans.}$$

Example 8.12 Two identical cylinders, each weighing 500 N are placed in a trough as shown in Fig. 8.14(a). Determine the reactions developed at contact points *A*, *B*, *C* and *D*. Assume all points of contact are smooth.

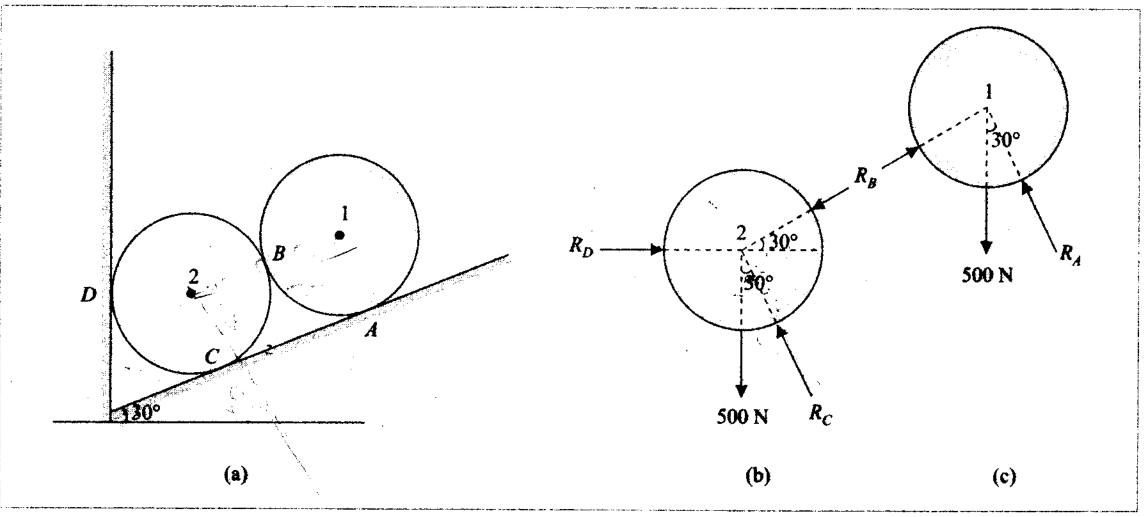


Fig. 8.14

Solution. Free body diagrams of two cylinders are as shown in Figs. 8.14(b) and (c).

Consider the equilibrium of cylinder 1. Since R_A is at right angles to the plane, if makes 60° to the horizontal, i.e. 30° to vertical. R_B is parallel to plane since the cylinders are identical. Thus R_A and R_B are at right angles to each other.

$\sum$ Force normal to plane = 0, gives

$R_A - 500 \cos 30 = 0$ or $\mathbf{R_A = 433\ N}$ **Ans.**

$\sum$ Force parallel to plane = 0, gives

$R_B - 500 \sin 30 = 0$ or $\mathbf{R_B = 250\ N}$ **Ans.**

Now, consider equilibrium of cylinder 2. R_D is horizontal since the line of contact is vertical. R_C is normal to the plane and R_B is parallel to the plane.

$$\sum F_V = 0 \rightarrow$$

$$-500 + R_C \cos 30 - R_B \sin 30 = 0$$

$$\therefore \quad R_C = \frac{500 + 250 \sin 30}{\cos 30} = \mathbf{721.7\ N} \quad \textbf{Ans.}$$

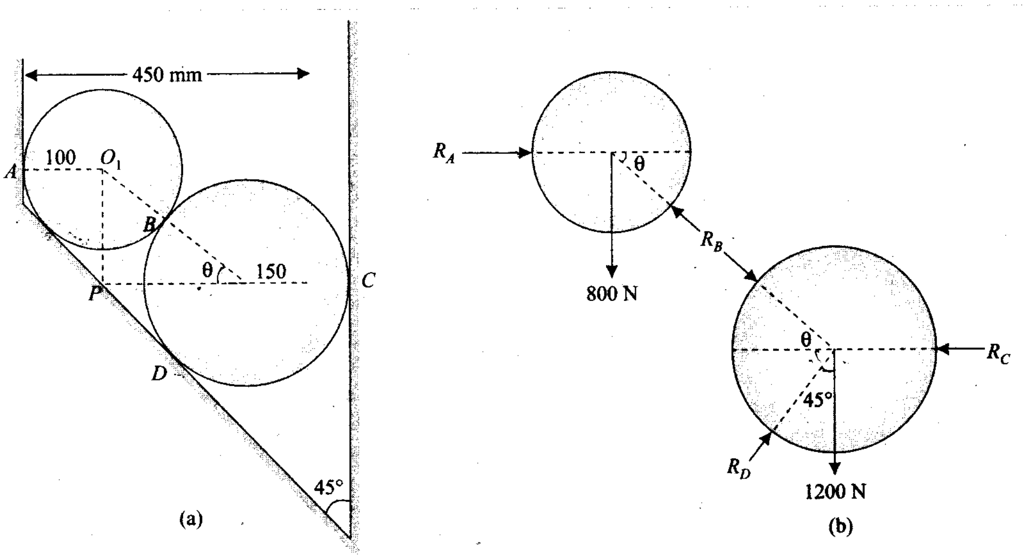
$$\sum F_H = 0 \rightarrow$$

$$R_D - R_C \sin 30 - R_B \cos 30 = 0$$

$$R_D = 721.7 \sin 30 + 250 \cos 30$$

$$\therefore \quad \mathbf{R_D = 577.4\ N} \quad \textbf{Ans.}$$

Example 8.13 Cylinder 1 of diameter 200 mm and cylinder 2 of diameter 300 mm are placed in a trough as shown in Fig. 8.15(a). If cylinder 1 weighs 800 N and cylinder 2 weighs 1200 N, determine the reactions developed at contact surfaces *A*, *B*, *C* and *D*. Assume all contact surfaces are smooth.



Solution. Since all contact surfaces are smooth reactions are at right angles to the contact surfaces, i.e. R_A and R_C are horizontal and R_D makes 45° to horizontal/vertical. R_A should be in the direction of O_1O_2. Let O_1O_2 make an angle θ with horizontal. Let O_1P and O_2P be vertical and horizontal. Then

$$\cos\theta = \frac{O_2P}{O_1O_2} = \frac{450-100-150}{100+150} = 0.8$$

$\therefore$ $$\theta = 36.87°$$

Consider the equilibrium of cylinder '1'

$$\sum F_V = 0 \rightarrow R_B \sin\theta - 800 = 0$$

$\therefore$ $$\mathbf{R_B = 1333.3\ N}$$ **Ans.**

$$\sum F_H = 0 \rightarrow R_A - R_B \cos\theta = 0$$

$\therefore$ $$R_A = 1333.3 \cos 36.87 = \mathbf{1066.7\ N}$$ **Ans.**

Now consider the equilibrium of cylinder 2,

$$\sum F_V = 0 \rightarrow R_D \cos 45 - R_B \sin\theta - 1200 = 0$$

$\therefore$ $$R_D = \frac{1333.3 \sin 36.87 + 1200}{\cos 45} = \mathbf{2828.4\ N}$$ **Ans.**

$$\sum F_H = 0 \rightarrow R_D \sin 45 + R_B \cos\theta - R_C = 0$$

$$2828.4 \sin 45 + 1333.3 \cos 36.87 - R_C = 0$$

$\therefore$ $$\mathbf{R_C = 3066.7\ N}$$ **Ans.**

Example 8.14 A 600 N cylinder is supported by the frame *BCD* as shown in Fig. 8.16(a). The frame is hinged at *D*. Determine the reactions developed at contact points *A*, *B*, *C* and *D*. Neglect the weight of frame and assume all contact surfaces are smooth.

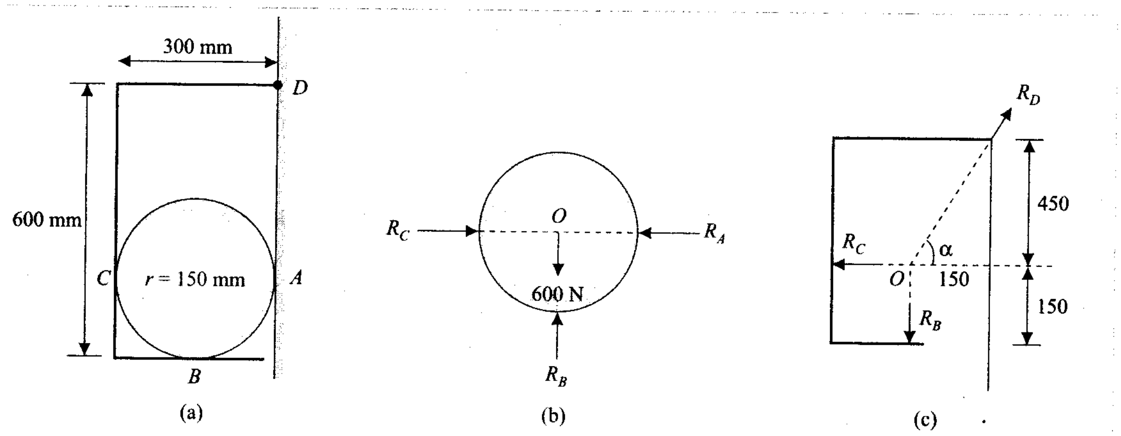


Fig. 8.16

Solution. Free body diagrams of the cylinder and the frame are shown in Figs. 8.16(b) and (c) respectively. Considering the equilibrium conditions of cylinder
we get,

$$\sum F_V = 0 \rightarrow \mathbf{R_B = 600\ N} \qquad \text{...(i) Ans.}$$

$$\sum F_H = 0 \rightarrow R_A = R_C \qquad \text{...(ii)}$$

Consider the equilibrium of the frame. As it is in equilibrium under the action of three forces only, they must be concurrent forces. In other words, the reaction at D has line of action along OD. Hence its inclination to horizontal is given by

$$\alpha = \tan^{-1}\frac{450}{150} = 71.56°$$

$$\therefore \quad \sum F_V = 0 \rightarrow R_D \sin\alpha = R_B = 600$$

$$\therefore \quad R_D = \frac{600}{\sin 71.56} = \mathbf{632.4\ N} \qquad \textbf{Ans.}$$

$$\sum F_H = 0 \rightarrow R_D \cos\alpha - R_C = 0$$

$$\therefore \quad R_C = 632.4 \cos 71.56 = \mathbf{200\ N} \qquad \textbf{Ans.}$$

Hence from eqn. (ii) $\quad R_A = \mathbf{200\ N}$ **Ans.**

Example 8.15 Cylinder A weighing 4000 N and cylinder B weighing 2000 N rest on smooth inclines as shown in Fig. 8.17(a). They are connected by a bar of negligible weight hinged to geometric centres of the cylinders by smooth pins. Find the force P to be applied as shown in the figure such that it will hold the system in the given position.

Solution. Figures 8.17(b) and (c) show the free body diagram of two cylinders. Applying Lami's theorem to the system of forces on cylinder A, we get

$$\frac{C}{\sin(180-60)} = \frac{4000}{\sin(60+90-15)}$$

$$\therefore \quad C = 4899\ \text{N}$$

Now consider equilibrium of cylinder B. Summation of forces parallel to the inclined plane = 0, gives

$$P\cos 15 + 2000\cos 45 - C\cos(45+15) = 0$$

$$\therefore \quad P = \frac{-2000\cos 45 + 4899\cos 60}{\cos 15} = \mathbf{1071.8\ N} \qquad \textbf{Ans.}$$

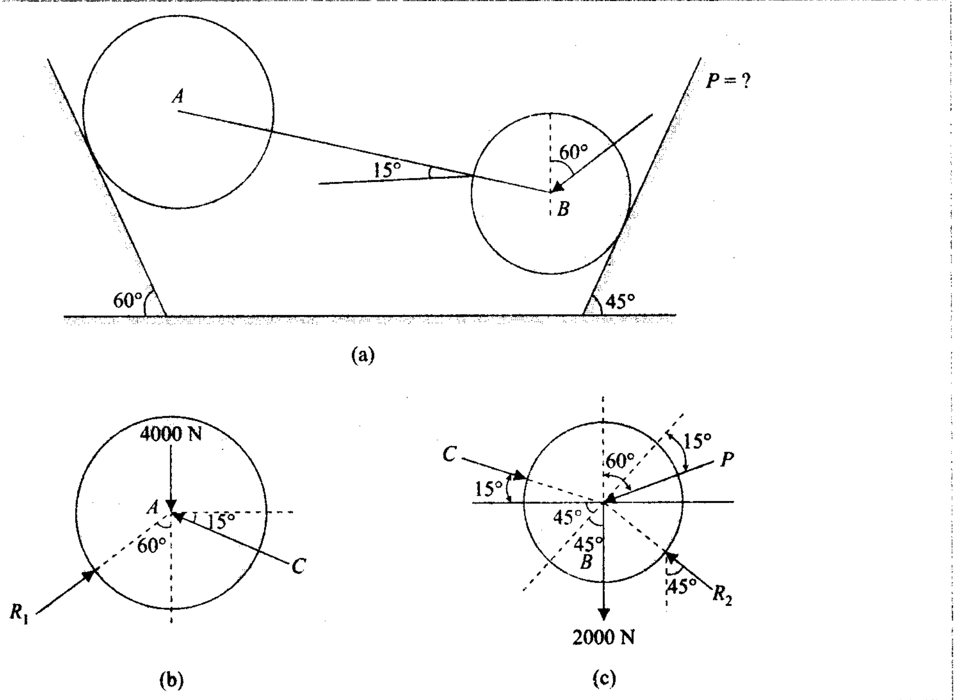


Fig. 8.17

Important Definitions

Free body diagram: The diagram showing the body freed from all other bodies in contact and showing all the forces acting on it, includes self weight and the reactions from the other bodies removed.

Lami's theorem: If a body is in equilibrium under the action of only three forces, each force is proportional to the sine of the angle between the other two forces.

Important Formulae

1. Equilibrant is same as the resultant in magnitude but its direction is opposite to that of the resultant.
2. According to Lami's theorem,

$$\frac{F_1}{\sin\alpha} = \frac{F_2}{\sin\beta} = \frac{F_3}{\sin\gamma}$$

   where α, β and γ are angles between F_2 - F_3, F_3 - F_1 and F_1 - F_2 respectively. This formula is to be used, if and only if the body is in equilibrium under the action of three forces only.
3. If a body is in equilibrium under the action of only three forces, they should be concurrent forces.

Problems for Exercise

1. A chord supported at A and B carries a load of 10 kN at D and a load of W at C as shown in Fig. 8.18. Find the value of W so that CD remain horizontal.

[**Ans.** $W = 30$ kN]

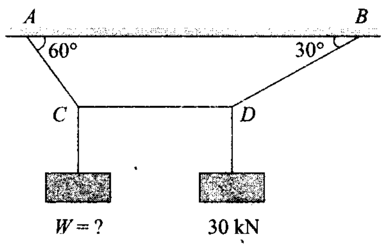


Fig. 8.18

2. Three bars, hinged at A and D pinned at B and C as shown in Fig. 8.19 form a four-linked mechanism. Determine the value of P that will prevent movement of bars.

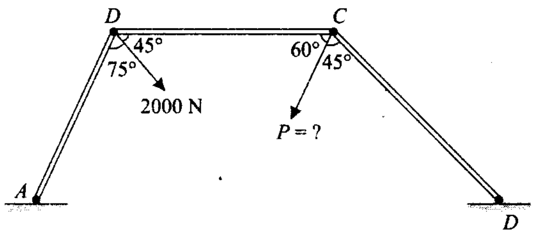


Fig. 8.19

3. Two smooth spheres each of radius 100 mm and weighing 100 N, rest in a horizontal channel having vertical walls, the distance between which is 360 mm. Find the reactions at the points of contact A, B, C and D as shown in Fig. 8.20.

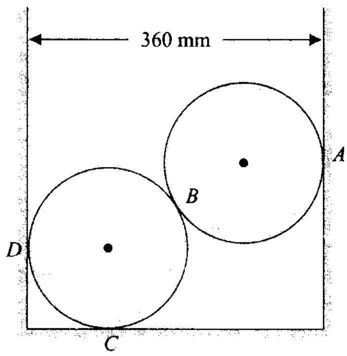


Fig. 8.20

4. Three cylinders are placed in a rectangular ditch as shown in Fig. 8.21. Neglecting friction, determine the reaction between cylinder *A* and the vertical wall. Weights and radii of the cylinder are as given below:

Cylinder	*Weight*	*Radius*
A	75 N	100 mm
B	200 N	150 mm
C	100 N	125 mm

[**Ans.** R = 400 N]

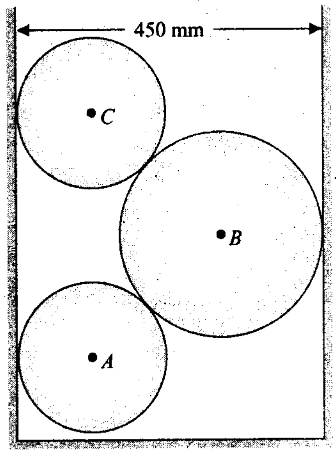


Fig. 8.21

5. Three spheres *A*, *B* and *C* having their diameters 500 mm, 500 mm and 800 mm, respectively are placed in a trench with smooth side walls and floor as shown in Fig. 8.22. The centre-to-centre distance of spheres *A* and *B* is 600 mm, The cylinders *A*, *B*, *C* weigh 4 kN, 4 kN and 8 kN respectively. Determine the reactions developed at contact points *P*, *Q*, *R* and *S*.

[**Ans.** R_P = 2.15 kN, R_Q = 7.44 kN, R_B = 7.03 kN, and R_S = 2.29 kN]

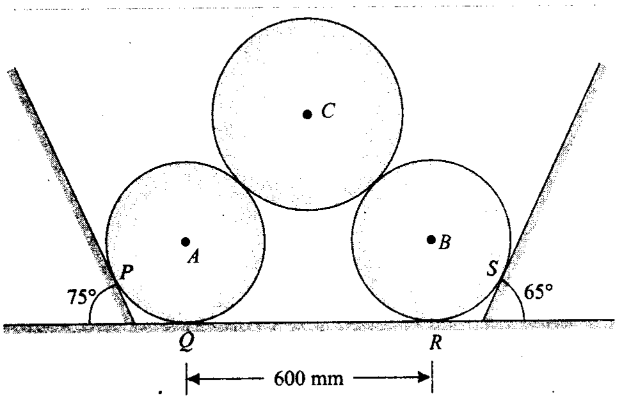


Fig. 8.22

6. Three smooth spheres A, B and C weighing 300 N, 600 N and 300 N respectively and having diameters 800 mm, 1200 mm and 800 mm respectively are placed in a trench as shown in Fig. 8.23. Determine the reactions developed at contact points P, Q, R and S.

[**Ans.** R_P = 61.24 N, R_Q = 631.6 N, R_R = 1095.2 N and R_S = 290.4 N]

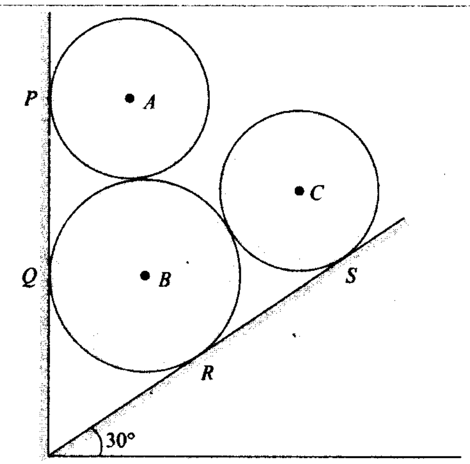


Fig. 8.23

CHAPTER 9

Equilibrium of Non-concurrent System of Forces and Support Reactions

In this chapter analysis of equilibrium conditions of non-concurrent system of forces is discussed. Equations of equilibrium are derived and used for finding the reactions in many engineering problems. Very common problem in this category of problems is finding the reactions in beams. This category of problems is dealt in detail.

9.1 EQUILIBRIUM EQUATIONS

A body is said to be in equilibrium when it does not have any translatory or rotational moment. This means when a body is in equilibrium under the action of coplanar non-concurrent forces the following equations are satisfied.

(i) The algebraic sum of the component of forces along any two mutually perpendicular direction is zero.

(ii) The algebraic sum of the moments of the forces about any point in the plane is zero.

Mathematically

$$R_x = \sum F_x = 0; \quad R_y = \sum F_y = 0 \quad \text{and} \quad \sum M_A = 0 \qquad \text{Eqn. (9.1)}$$

Referring to Fig. 9.1, A, B, C, are three points which are not collinear. Let R be the resultant of the system of forces on the body. Then,

$\Sigma M_A = 0$, means moment of all the forces about point A is zero. In other words, resultant passes through point A.

$\Sigma M_B = 0$, then B is also the point through which resultant passes. In other words, AB is the line of action of the resultant. If point C is not collinear with AB, then $\Sigma M_C = 0$ means, R itself is zero, since R cannot pass through C. Thus if A, B, C are not collinear, the following conditions are also necessary and sufficient conditions of equilibrium:

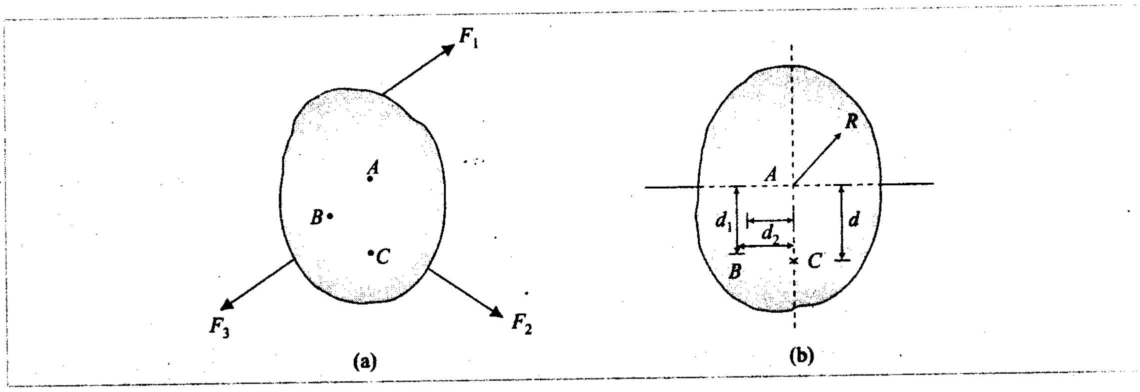


Fig. 9.1

$$\sum M_A = 0, \quad \sum M_B = 0 \quad \text{and} \quad \sum M_C = 0 \qquad \text{Eqn. (9.2)}$$

The equilibrium conditions (eqn. 9.2) are not independent of conditions 9.1. Two of them are common to the two sets. Referring to Fig. 9.1(b), R can be resolved into its components perpendicular to and parallel to AC.

Then, $\sum M_C = 0$

means $R_x\, d = 0$

i.e., $R_x = 0$

Now $\sum M_B = 0$

means $R_x\, d_1 + R_y\, d_2 = 0$

i.e., $R_y\, d_2 = 0$, since $d_2 \neq 0$

$\therefore$ $R_y = 0$

Thus $\sum M_C = 0$ is equivalent to $R_x = 0$ and

$\sum M_B = 0$ is equivalent to $R_y = 0$. Thus eqn. (9.1) is not independent of eqn. (9.2).

In fact, any of the following set of equilibrium equations can be used:

1. $R_x = \Sigma F_x = 0,\ R_y = \Sigma F_y = 0$ and $M_A = 0$
2. If line AB is not in y-direction,
   $R_y = \Sigma F_y = 0,\ \Sigma M_A = 0$ and $\Sigma M_B = 0$
3. If line AB is not in x-direction,
   $R_x = \Sigma F_x = 0,\ \Sigma M_A = 0$ and $\Sigma M_B = 0$
4. If A, B and C are non-collinear
   $\Sigma M_A = 0,\ \Sigma M_B = 0$ and $\Sigma M_C = 0$

Eqn. (9.3)

Now we are in a position to prove that if a system is in equilibrium under the action of only three forces, they must be concurrent.

Let F_1, F_2, and F_3 be the three forces acting on a body and the system is in equilibrium. Then, if A is the point of intersection of forces F_1 and F_2, the equilibrium condition gives

$$\sum M_A = 0$$

i.e., $$F_3 d = 0$$

where d is the distance of line of action of F_3 from A. Since F_3 is not zero,

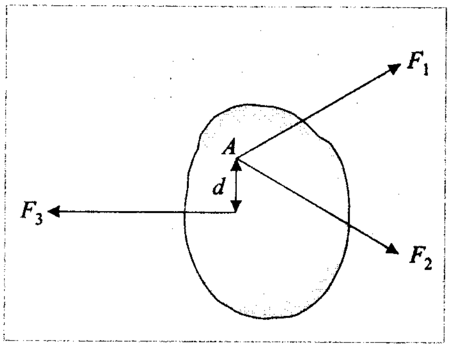


Fig. 9.2

$$d = 0$$

In other words, the third force F_3 also should pass through A. Hence the proposition is proved.

Example 9.1 The 12 m boom AB weighs 10 kN, the distance of the centre of gravity G being 6 m from hinge A. For the position shown, determine the tension T in the cable BC and the reaction at hinge A. [Ref. Fig. 9.3(a)].

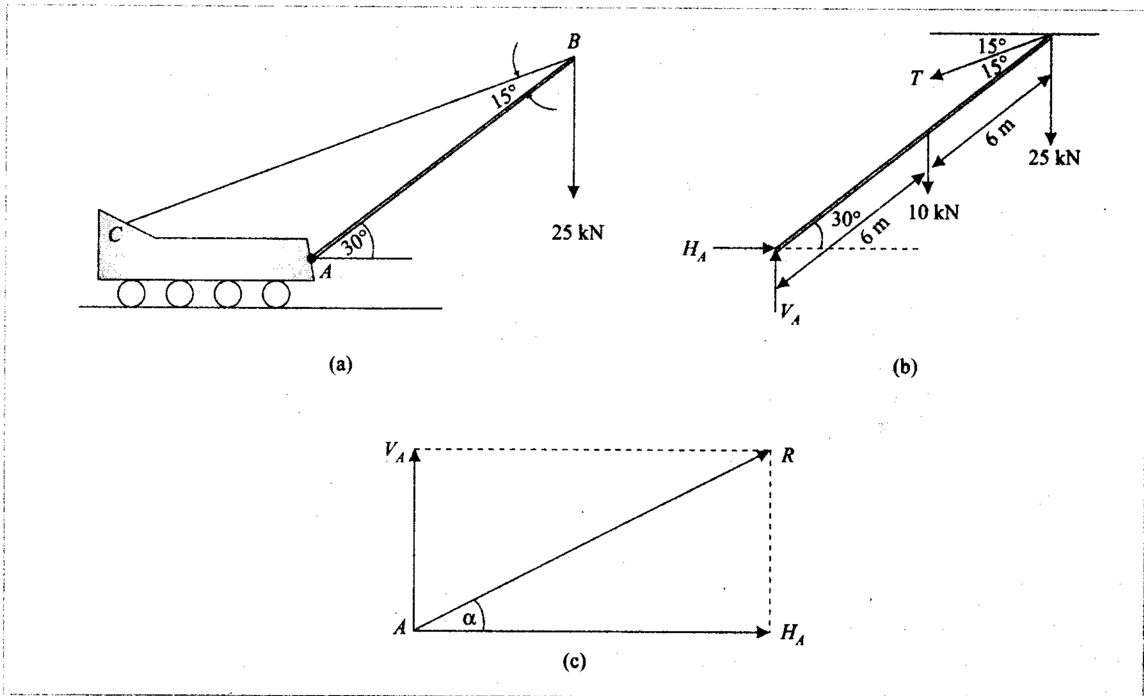


Fig. 9.3

Solution. The free body diagram of the boom is shown in Fig. 9.3(b). Since A is a hinged end,

$$\sum M_A = 0 \rightarrow$$

$$-T \times 12 \sin 15 + 25 \times 12 \cos 30 + 10 \times 6 \cos 30 = 0$$

$\therefore$ $$T = \mathbf{100.38\ kN}$$ **Ans.**

$$\sum F_H = 0 \rightarrow$$

$$H_A - T \cos 15 = 0$$

$\therefore$ $$H_A = 100.38 \cos 15 = 96.992 \text{ kN.}$$

$$\sum F_V = 0 \rightarrow$$

$$V_A - 10 - 25 - T \sin 15 = 0$$

$$V_A = 60.980 \text{ kN}$$

$$R_A = \sqrt{96.992^2 + 60.980^2} = \mathbf{114.569\ kN}$$ **Ans.**

$$\alpha = \tan^{-1} \frac{60.980}{96.992} = \mathbf{32.16°} \text{ as shown in Fig. 9.3(c).}$$ **Ans.**

Example 9.2 A ladder weighing 100 N is to be kept in the position shown in Fig. 9.4(a), resting on a smooth floor and leaning on a smooth wall. Determine the horizontal force required at floor level to prevent it from slipping when a man weighing 700 N is at 2 m above floor level.

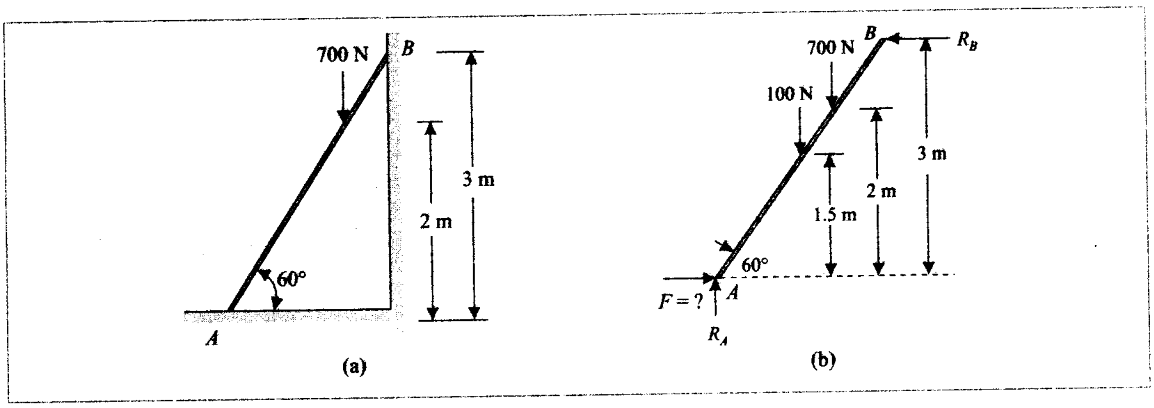


Fig. 9.4

Solution. Free body diagram of the ladder is as shown in Fig. 9.4(b). R_A is vertical and R_B is horizontal because the surfaces of contact are smooth. Self weight of 100 N acts through centre point of ladder in vertical direction. Let F be the horizontal force required to be applied to prevent slipping.

Then,

$$\sum M_A = 0 \rightarrow \quad -R_B \times 3 + 700 \times 2 \cot 60 + 100 \times 1.5 \cot 60 = 0$$

$\therefore$ $$R_B = 298.3 \text{ N}$$

$$\sum F_H = 0 \rightarrow$$

$$F - R_B = 0$$

$\therefore$ $$\mathbf{F = R_B = 298{,}3\ N}$$ **Ans.**

Example. 9.3 In the above ladder problem, if the horizontal force F is to be applied at a height of 1 m above the ground level, how much should F be?

Solution. Figure 9.5 shows the free body diagram of the ladder for this case.

$$\sum F_H = 0 \qquad \rightarrow F - R_B = 0 \text{ or } F = R_B \qquad ...(i)$$

$$\sum M_A = 0 \rightarrow$$

$$-R_B \times 3 + 700 \times 2 \cot 60 + 100 \times 1.5 \cot 60 + F \times 1 = 0 \quad ...(ii)$$

i.e.,

$$3F - F = (700 \times 2 + 100 \times 1.5) \cot 60, \text{ since } R_B = F.$$

$\therefore$ $$\mathbf{F = 459\ N}$$ **Ans.**

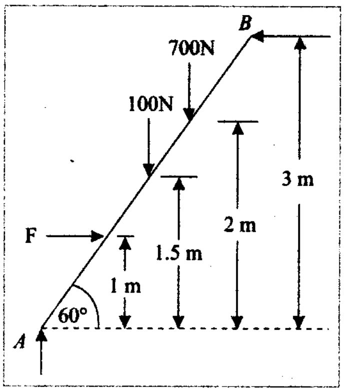


Fig. 9.5

Example 9.4 A roller weighing 2000 N rests on an inclined bar weighing 800 N as shown in Fig. 9.6(a). Assuming weight of bar AB is negligible, determine the reactions developed at supports C and D.

Solution. Free body diagrams of roller and bar CD are as shown in Figs. 9.6(b) and (c). [**Note:** R_D is vertical since at D roller support is horizontal. At C reaction can be in any direction, since it is hinged support. Hence components of reaction at C are taken as H_C and V_C].

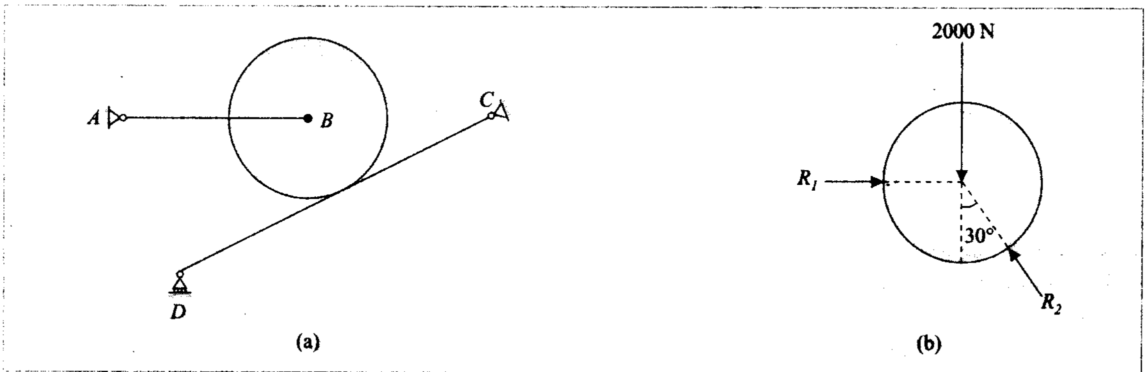


Fig. 9.6 (contd.)

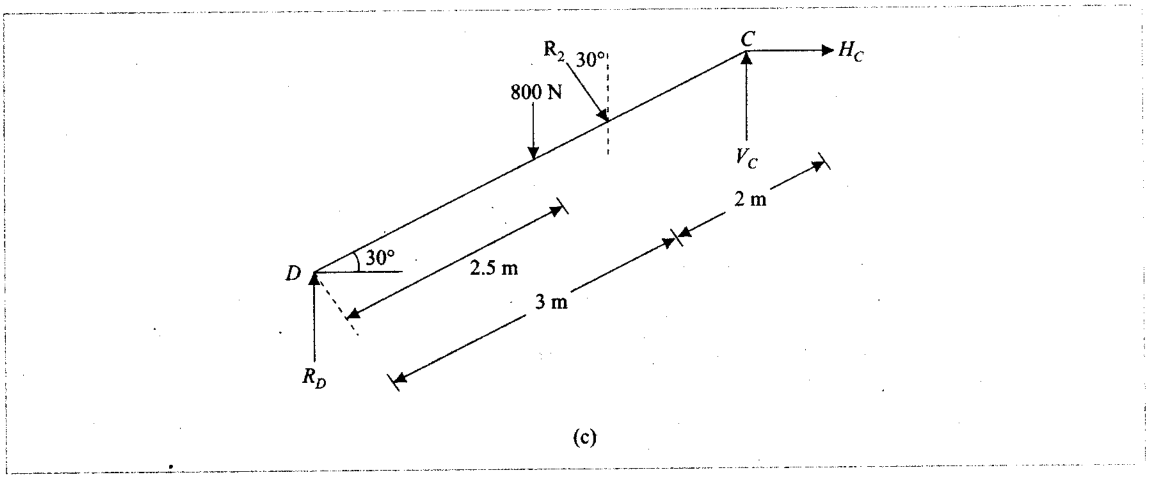


Fig. 9.6

For roller,

$\sum F_V = 0 \rightarrow \qquad R_2 \cos 30 = 2000$

$\therefore \qquad R_2 = 2309.4$ N

Considering the equilibrium of the bar *CD*,

$\sum M_C = 0 \rightarrow \qquad R_D\, 5 \cos 30 - 800 \times 2.5 \cos 30 - R_2 \times 2 = 0$

$$R_D = \frac{800 \times 2.5 \cos 30 + 2309.4 \times 2}{5 \cos 30}$$

i.e., $\mathbf{R_D = 1466.7}$ **N** **Ans.**

$\sum F_V = 0 \rightarrow$

$$R_D - 800 - 2309.4 \cos 30 + V_C = 0$$

$\therefore \qquad V_C = 800 + 2309.4 \cos 30 - 1466.7 = \mathbf{1333.3}$ **N** **Ans.**

$\sum F_H = 0 \rightarrow$

$$R_2 \sin 30 - H_C = 0$$

$\therefore \qquad H_C = 2309.4 \sin 30 = \mathbf{1154.7}$ **N** **Ans.**

Example 9.5 A cable car used for carrying materials in a hydroelectric project is at rest on a track formed at an angle of 60° to horizontal. The gross weight of the car and its load is 60 kN and its centre is at a point 800 mm from the track half way between the axles. The car is held by a cable as shown in Fig. 9.7. The axles of the car are at a distance 1.2 m. Find the tension in the cables and reaction at each of the axles neglecting friction of the track.

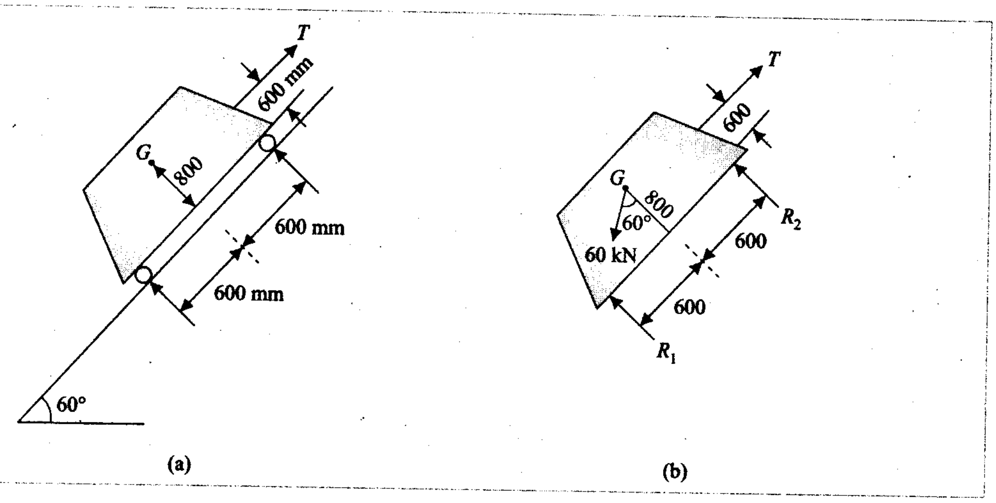


Fig. 9.7

Solution. The free body diagram of the car is shown in Fig. 9.7(b). The self weight is vertical and hence makes 60° to the normal to the inclined plane. Let T be the tension in the rope which is parallel to track.

Now, Σ Forces parallel to track = 0, gives,

$$T - 60 \sin 60 = 0$$

or $$\mathbf{T = 51.961 \text{ kN}}$$ **Ans.**

Applying moment equilibrium condition about upper axle point on track, we get

$$R_1 \times 1200 + T \times 600 - 60 \sin 60 \times 800 - 60 \cos 60 \times 600 = 0$$

∴ $$\mathbf{R_1 = 23.660 \text{ kN.}}$$ **Ans.**

Σ Forces normal to the plane = 0, gives,

$$R_1 + R_2 - 60 \cos 60 = 0$$

∴ $$R_2 = 60 \cos 60 - R_1 = 60 \cos 60 - 23.660$$

i.e., $$\mathbf{R_2 = 6.34 \text{ kN}}$$ **Ans.**

Example 9.6 A hollow right circular cylinder of diameter 1600 mm is open at both ends and rests on a smooth horizontal plane as shown in Fig. 9.8(a). A sphere of radius 600 mm weighing 3 kN is first put in the cylinder and then a sphere weighing 1 kN and of radius 400 mm was placed. Neglecting the friction, find the minimum weight of hollow cylinder so that it will not tip over.

Solution. Join the centre of spheres O_1 and O_2 and drop O_1D perpendicular to horizontal through O_2.

Now, $$O_1O_2 = 400 + 600 = 1000 \text{ mm}$$

$$O_2D = 1600 - 400 - 600 = 600 \text{ mm}$$

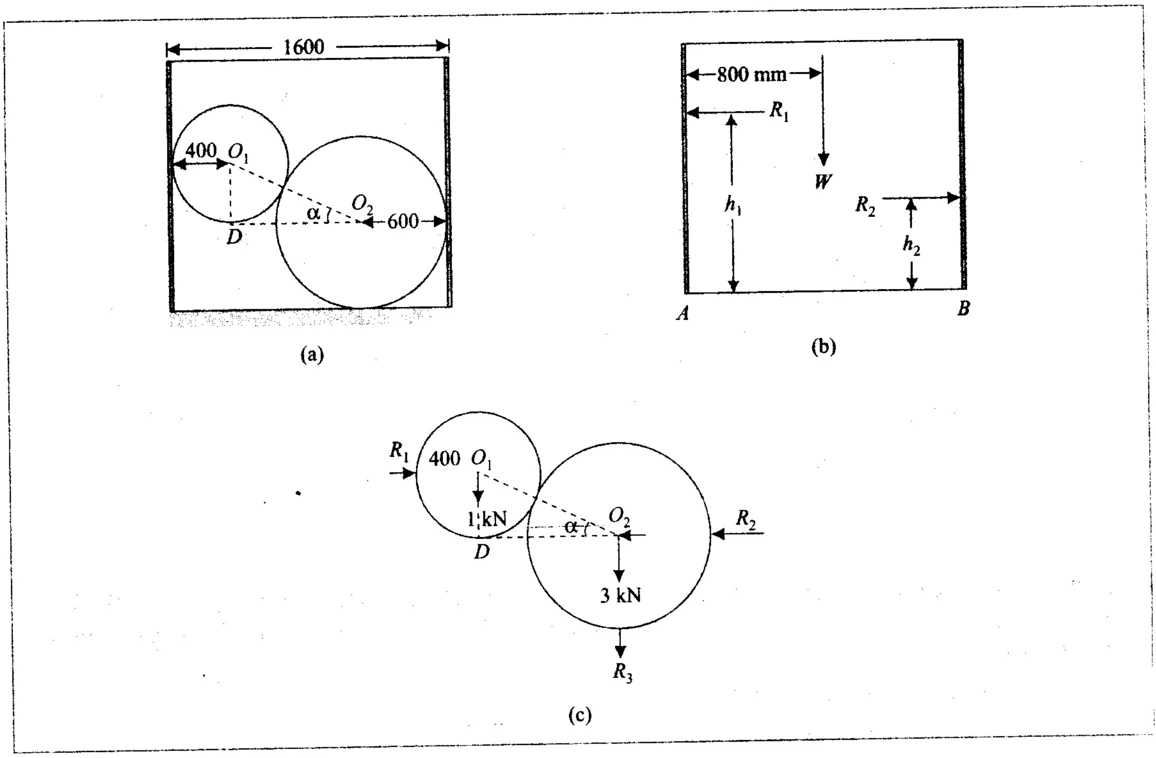


Fig. 9.8

If α is the inclination of O_2O_1 to horizontal,

$$\alpha = \cos^{-1}\frac{600}{1000}$$

$$= 53.13°$$

Free body diagrams of cylinder and spheres are shown in Figs. 9.8(b) and (c). Considering the equilibrium of the spheres,

$\sum M_{O_2} = 0$, gives,

$$R_1 \times O_1O_2 \sin \alpha - 1 \times O_2D = 0$$

i.e.,

$$R_1 \times 1000 \sin 53.13 = 1 \times 600$$

∴

$$R_1 = 0.75 \text{ kN.}$$

$\sum F_H = 0$, gives,

$$R_2 = R_1 = 0.75 \text{ kN.}$$

$\sum F_V = 0$, gives,

$$R_3 = 1 + 3 = 4 \text{ kN.}$$

Now consider the equilibrium of the cylinder. Let the minimum weight be W. Tipping will take place over point A. Hence at this stage there will not be any reaction at point B. Hence,

$\sum M_A = 0$, gives,

$$R_1 h_1 - R_2 h_2 - W \times 800 = 0$$

$$R_1 (h_1 - h_2) = W \times 800, \qquad \text{since } R_1 = R_2.$$

i.e.,
$$R_1 \times O_1 D = W \times 800$$

$\therefore$
$$W = \frac{0.75 \times 1000 \sin 53.13}{800} = \mathbf{0.75\ kN}$$ **Ans.**

Example 9.7 A 500 N cylinder of 1 m diameter is loaded between the cross-pieces which make an angle of 60° with each other and are pinned at C. Determine the tension in the horizontal rope DE, assuming a smooth floor [Ref. Fig. 9.9(a)].

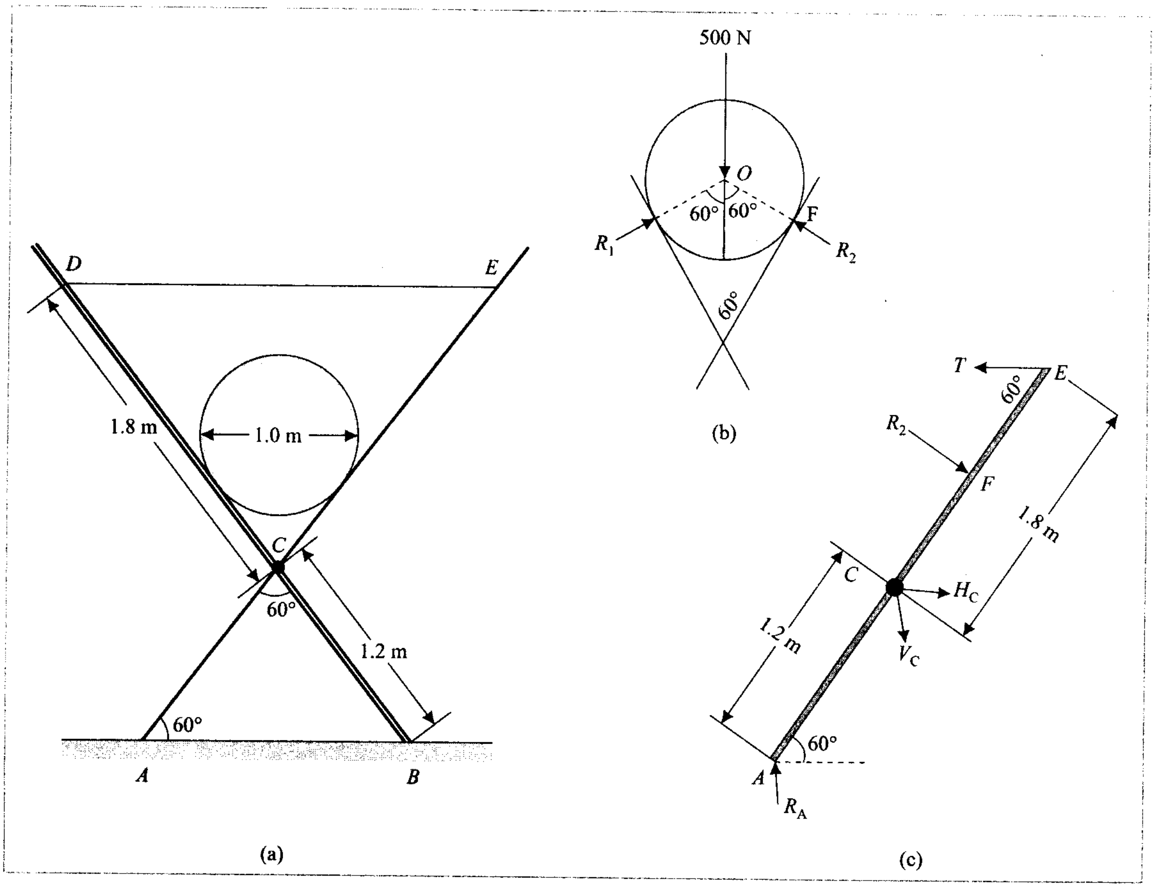


Fig. 9.9

Solution. Consider the equilibrium of the entire system. Due to symmetry, $R_A = R_B$

$\sum F_V = 0 \rightarrow$

$$R_A + R_B = 500$$

$$\therefore \quad R_A = R_B = 250 \text{ N}$$

Now consider the equilibrium of the cylinder [Ref. Fig. 9.9(b)]. The members AE and BD make 60° with the horizontal. Hence the reactions R_2 and R_1 make 60° with the vertical.

$\sum F_H = 0$, gives,

$$R_1 \sin 60 - R_2 \sin 60 = 0$$

$$\therefore \quad R_1 = R_2$$

$\sum F_V = 0$, gives,

$$R_1 \cos 60 + R_2 \cos 60 - 500 = 0$$

Since, $R_1 = R_2$, we get

$$2R_1 \cos 60 = 500$$

i.e.,

$$R_1 = 500 \text{ N}$$

Hence

$$R_2 = 500 \text{ N}$$

Now, consider the equilibrium of the member AE. Its free body diagram is shown in Fig. 9.9(c).

$\sum M_C = 0 \rightarrow$

$$T \sin 60 \times 1.8 - R_2 \times CF - R_A \times 1.2 \cos 60 = 0$$

Now

$$CF = OF \cot 30° = 0.5 \cot 30° = 0.866 \text{ m}$$

$$\therefore \quad T = \frac{500 \times 0.866 + 250 \times 1.2 \times 0.5}{1.8 \sin 60} = \mathbf{374.0 \text{ N}} \qquad \textbf{Ans.}$$

Note: If the reactions on the pin at C are required, the remaining two equations, namely, $\Sigma F_x = 0$ and ΣF_y, are to be considered.

9.2 APPLICATIONS TO BEAM PROBLEMS

A beam is a structural element which has one dimension (length) is considerably larger than the other two dimensions in the cross-section and is supported at few points. It is subjected to lateral loads. Due to applied loads, reactions develop at supports and the system of forces consisting of applied loads, self weight (many times neglected) and reactions keep the beam in equilibrium. The forces constitute a system of coplanar non-concurrent system in equilibrium. If the support reactions can be determined using the equations of equilibrium only, then the beam is said to be *statically determinate*. In this article, types of supports, types of beams and types of loading are explained first and then the method of finding reactions developed at the supports of determinate beams is illustrated.

9.3 TYPES OF SUPPORTS

Various types of supports and reactions developed at those supports are listed below:

Simple Support

The end of the beam rests simply on a rigid support as shown in Fig. 9.10. In the idealised simple support there is no resistance to the force in the direction of the support. Hence the reaction is always normal to the support. There will not be any restraint from the support for the rotation of the end of the beam. In other words, there is no moment resistance at support.

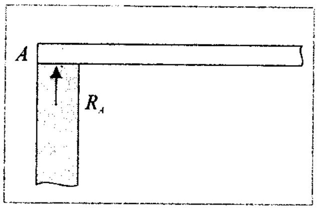


Fig. 9.10 Simple support

Roller Support

In this case, beam end is supported on rollers. In such cases, reaction is always normal to the support, since rollers are free to roll along the supports. In idealised condition rolling friction is neglected and hence there is no resistance in the line of support. The ends are free to rotate also. Hence there is no resistance to moment. Many mechanical components are provided with roller supports which roll between girders. In such cases reaction can be normal to girder in either direction (Ref. Fig. 9.11).

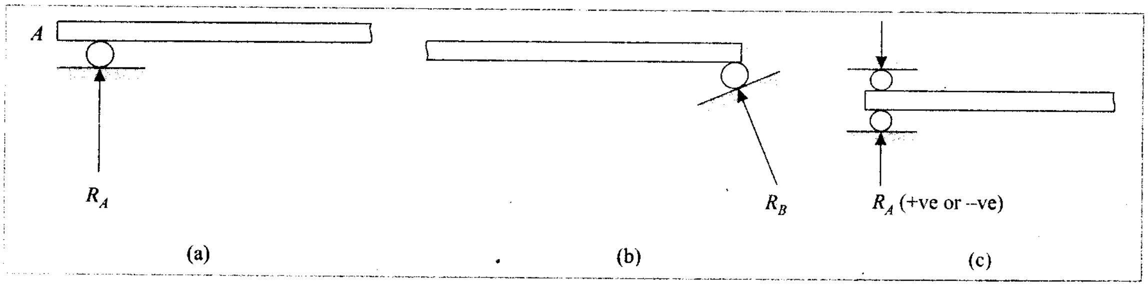


Fig. 9.11 Roller supports

Hinged or Pinned Support

In such cases, the position of the end of the beam fixed to the end is free to rotate. This idealised condition can be achieved by using mechanical devices. A typical hinged end condition is shown in Fig. 9.12. At such supports the reaction can be in any direction which is usually represented by its components in two mutually perpendicular directions. This type of supports does not provide any resistance to moment; in other words it permits rotation freely at the end.

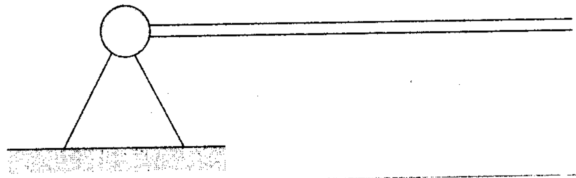


Fig. 9.12 Hinged or pinned support

Fixed Support

At fixed support the end of the beam is neither permitted to move in any direction nor allowed to rotate. Hence support reactions are a force in any direction and the resisting moment (Ref. Fig. 9.13). Reacting force in any direction is conveniently represented by its components in two mutually perpendicular directions. This end condition may be achieved by taking the end of the beam considerably inside the support or by specially designing the brackets to resist the movement and rotation.

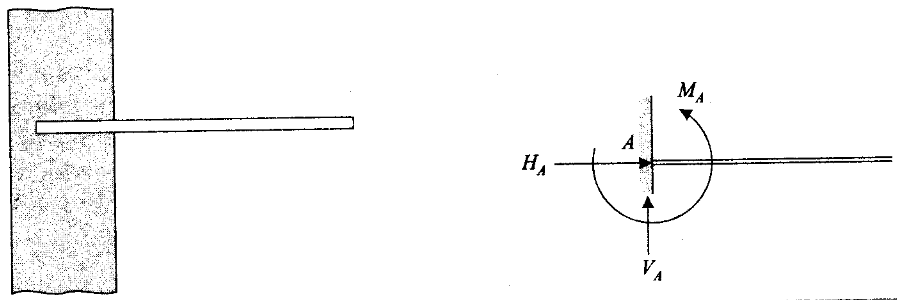


Fig. 9.13 Fixed supports

9.4 TYPES OF BEAMS

Depending upon the types of supports, beams may be classified as the following:

(i) Cantilever
(ii) Simply supported
(iii) One end hinged and other on roller
(iv) Overhanging
(v) Both ends hinged
(vi) Propped cantilever and
(vii) Continuous.

(i) **Cantilever:** If a beam is fixed at one end and free at other end, it is called a cantilever beam. In this there are three reaction components at fixed end [V_A, H_A, M_A as shown in Fig. 9.14] and no reaction component at free end.

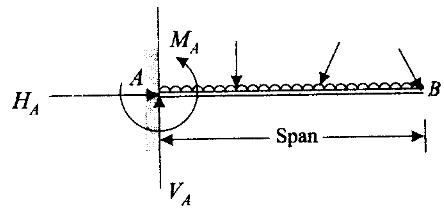


Fig. 9.14 Cantilever beam

(ii) **Simply supported beam:** In this type of beam both ends are simply supported as shown in Fig. 9.15. There is one reaction component at each end [R_A and R_B]. They act at right angles to the support. This type of beam can resist forces normal to the beam axis. In other words, the equilibrium condition that summation of forces parallel to axis equal to zero is satisfied automatically by the loading condition. Hence two equations of equilibrium are available.

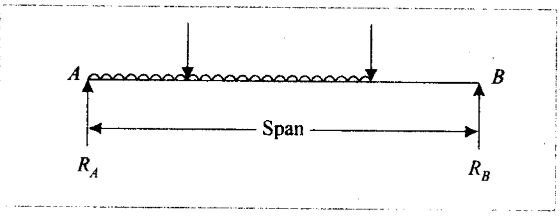


Fig. 9.15 Simply supported beam

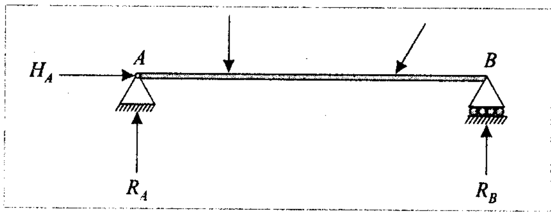


Fig. 9.16 One end hinged and other on roller

(iii) **One end hinged and the other on roller:** As the name suggests one end of the beam is hinged and the other end is on roller [Ref. Fig. 9.16]. At hinged end reaction can be in any direction and at roller end it is at right angles to the roller support. The hinged end reaction at any direction can be represented by its two components, perpendicular to each other. Thus the reaction components for such beam shown in Fig. 9.16 are V_A, H_A and R_B.

(iv) **Overhanging beam:** If a beam is projecting beyond the support/supports, it is called an overhanging beam [Fig. 9.17]. The overhang may be on only one side or may be on both sides.

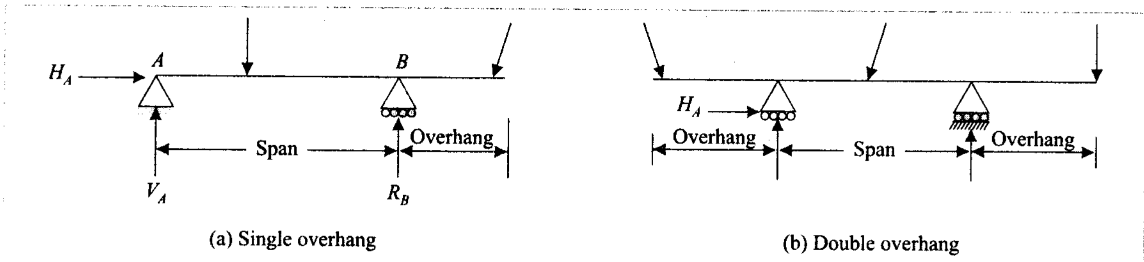


Fig. 9.17 Overhanging beams

(v) **Both ends hinged:** As name suggests both ends of the beam are hinged. There are two reaction components at each end, hence total reaction components are four.

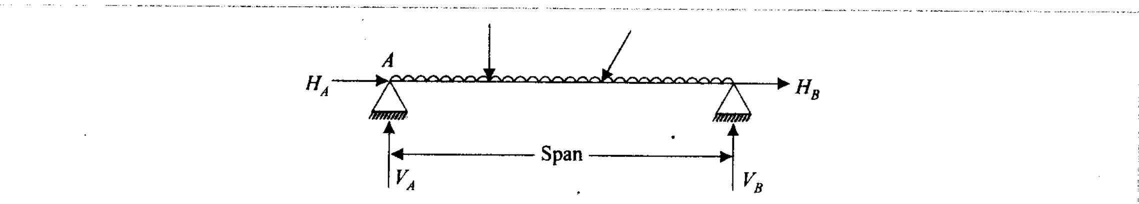


Fig. 9.18 Both ends hinged beam

(vi) **Propped cantilever:** In this type of beam one end of the beam is fixed and the other end is simply supported or is on rollers. It has four reaction components (V_A, H_A, M_A & R_B) as shown in Fig. 9.19.

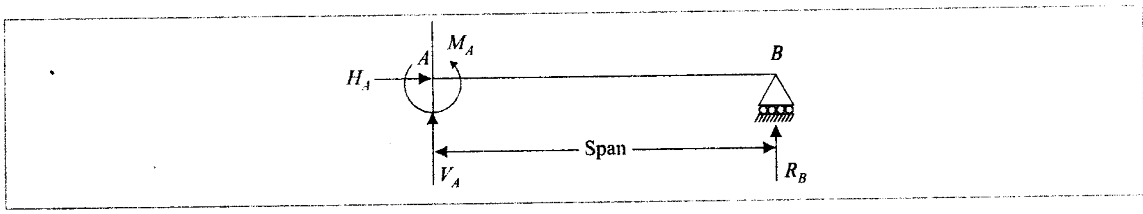


Fig. 9.19 Propped cantilever

(vii) **Continuous Beam:** A beam having three or more supports is called continuous beam. In such beams three or more reaction components exist [Ref. Fig. 9.20].

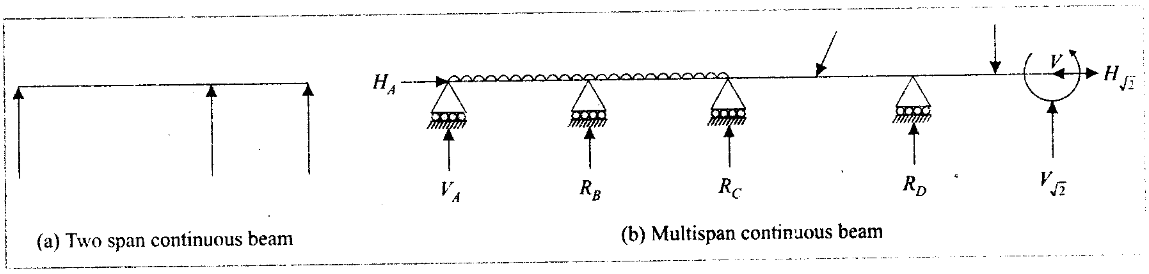


Fig. 9.20 Continuous beams

In cantilever beams, simply supported beams, one end hinged and the other on roller and overhanging beams, number of unknown reactions are equal to the number of independent equations of equilibrium. Hence for any given loading, reactions can be determined by using equations of equilibrium only. Therefore such beams are known as **statically determinate beams**. Thus we can define *statically determinate beams as those beams in which all reaction components can be found using the equations of equilibrium only.*

In case of beams with both ends hinged, propped cantilever and continuous beams, number of unknown reaction components are more than the number of available equations of equilibrium. This type of beams are known as **statically indeterminate beams**.

9.5 TYPES OF LOADING

Usual type of loadings on the beams are discussed below:

(i) **Concentrated Loads:** If a load is acting over a very small length compared to span of the beam, it is approximated as point load. It is represented by an arrow as shown in Fig. 9.21.

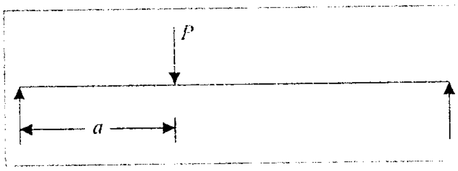


Fig. 9.21 Concentrated load

(ii) **Uniformly Distributed Load (UDL):** A load which has got same intensity over a considerable length is called uniformly distributed load. It is shown as indicated in Fig. 9.22(a) or as in Fig. 9.22(b).

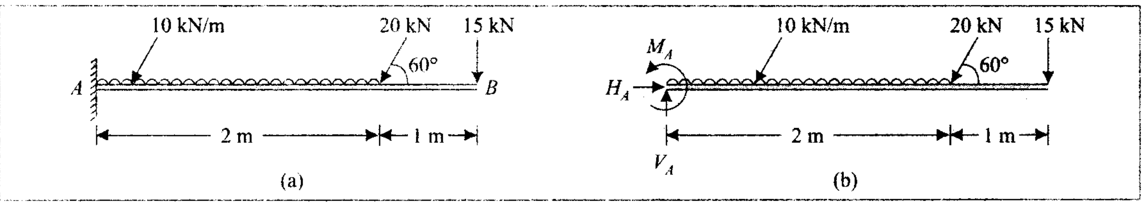


Fig. 9.22 Uniformly distributed load (UDL)

When equilibrium of entire beam is to be considered, this loading is imagined as total load (intensity × length) acting at the middle or length.

(iii) **Uniformly Varying Load:** If the intensity of the load increases linearly along length, it is called linearly varying load (Fig. 9.23). In the Fig. 9.23(a) load varies from C to D with zero intensity at C and 20 kN/m intensity at D. In the load diagram shown, the ordinate represents the intensity of load and the abscissa represents the position of the load on the beam. Hence if intensity at one end is zero, load diagram takes the shape of a triangle. Therefore it may be called as triangular load. It may be noted that the area of the triangle represents the total load and the centroid of the triangle represents the centre of gravity of the load. Thus total load in Fig. 9.23(a) is

$$\frac{1}{2} \times 3 \times 20 = 30 \text{ kN}$$

and the centre of gravity of this loading is at $\frac{1}{3} \times 3 = 1$ m from D. For finding the reaction we can imagine that the given load is equivalent to 30 kN acting at $1 + 3 - 1 = 3$ m from end A.

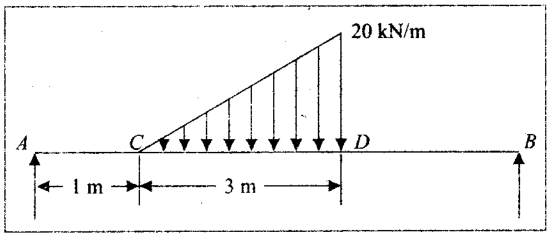


Fig. 9.23 Uniformly varying load

(iv) **General Loading:** Figure 9.24 shows the case of general loading. Here also ordinate represents the intensity of loading and abscissa represents the position of the loading. For simplicity, in the analysis such loadings are replaced by a set of equivalent concentrated loads.

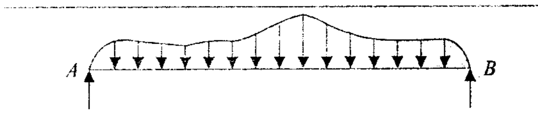


Fig. 9.24 General loading

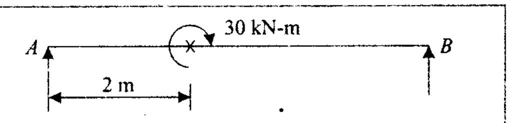


Fig. 9.25 External moment

(v) **External Moment:** A beam may be subjected to external moments at certain points. In Fig. 9.25 the beam is subjected to a clockwise moment of 30 kN-m at a distance 2 m from the left support.

9.6 FINDING SUPPORT REACTIONS

To resist the applied loads reactions develop at supports of the beam. Reactions are self adjusting forces which will keep the beams in equilibrium. Hence the equations of equilibrium may be written for the system of forces consisting of reactions and the loads. Solutions of these equations give the unknown reactions. The procedure for finding the reactions in statically determinate beams is illustrated with the examples below:

Example 9.8 Determine the reactions developed in the cantilever beam shown in Fig. 9.26a.

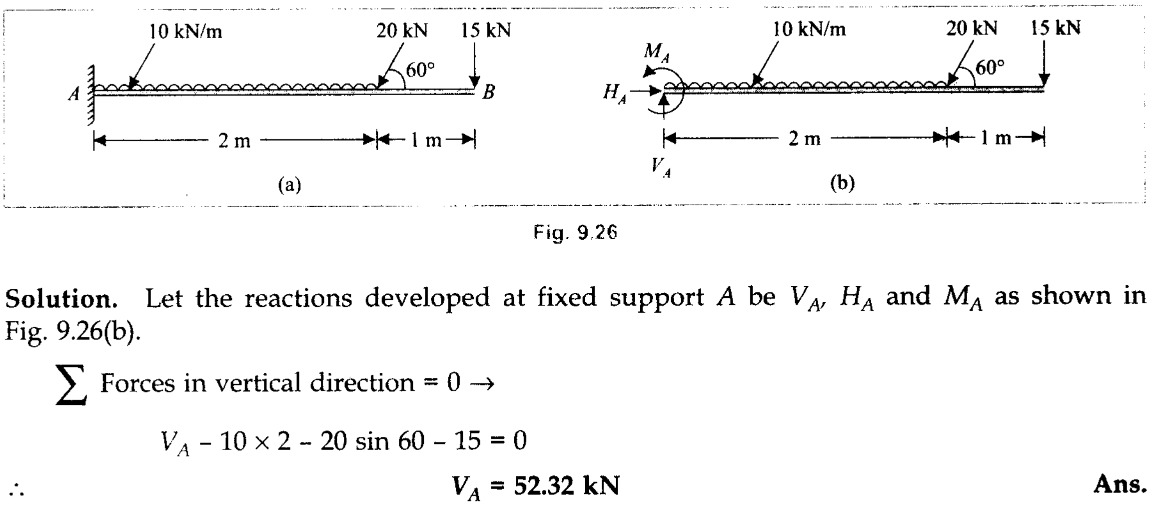


Fig. 9.26

Solution. Let the reactions developed at fixed support A be V_A, H_A and M_A as shown in Fig. 9.26(b).

$\sum$ Forces in vertical direction = 0 →

$$V_A - 10 \times 2 - 20 \sin 60 - 15 = 0$$

$\therefore$ $\quad V_A = \mathbf{52.32\ kN}$ **Ans.**

$\sum$ Forces in horizontal direction = 0 →

$$H_A - 20 \cos 60 = 0 \quad \therefore\ H_A = \mathbf{10\ kN}$$ **Ans.**

$\sum$ Moments about A = 0 →

$$-M_A + 10 \times 2 \times 1 + 20 \sin 60 \times 2 + 15 \times 3 = 0$$

$\therefore$ $\quad M_A = \mathbf{99.64\ kN\text{-}m}$ **Ans.**

Note: *UDL* is treated as a load 10 × 2 kN acting at its centre of gravity which is at 1 m from *A*.]

Example 9.9 Determine the reactions developed in the cantilever beam shown in Fig. 9.27(a).

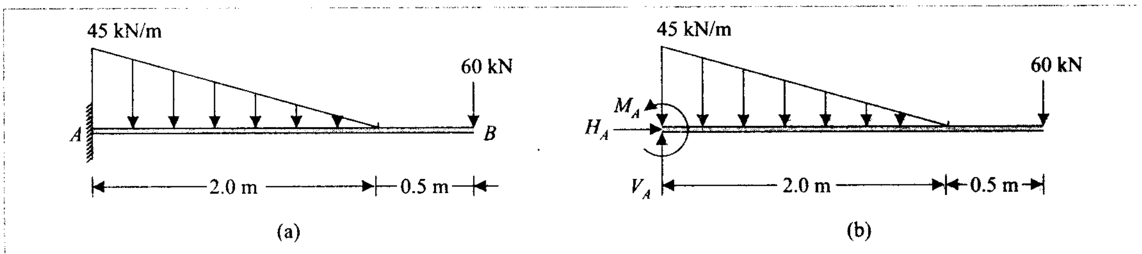


Fig. 9.27

Solution. Let the reactions developed at the fixed end A be V_A, H_A and M_A as shown in Fig. 9.27(b).

$$\sum V_A = 0 \rightarrow$$

$$V_A - \frac{1}{2} \times 45 \times 2 - 60 = 0$$

$$\mathbf{V_A = 105\ kN} \qquad \textbf{Ans.}$$

$$\sum H = 0 \rightarrow$$

$$H_A - 0 = 0 \quad \therefore H_A = 0$$

$$\sum M_A = 0 \rightarrow$$

$$-M_A + \frac{1}{2} \times 45 \times 2 \times \frac{2}{3} + 60 \times 2.5 = 0$$

$$\therefore \quad \mathbf{M_A = 180\ kN\text{-}m} \qquad \textbf{Ans.}$$

[**Note:** Total triangular load = $\frac{1}{2} \times 45 \times 2 = 45$ kN and its centroid is at $\frac{1}{3} \times 2$ m from end A].

Example 9.10 Determine the reaction developed in the simply supported beam in Fig. 9.28.

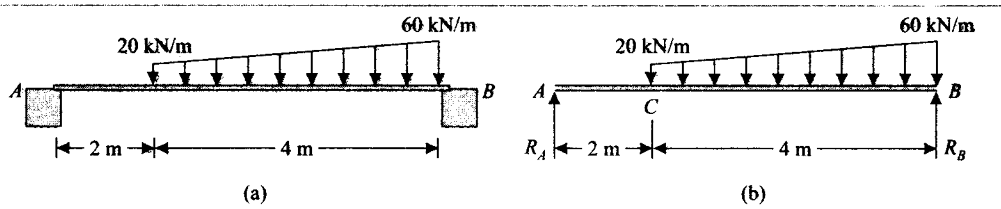


Fig. 9.28

Solution. Let R_A and R_B be the reactions developed at the simply supported ends A and B. The uniformly varying load may be split into a uniformly distributed load of 20 kN/m intensity and a triangular load of intensity zero at point C and intensity 60 – 20 = 40 kN/m at B. Then

$$\sum M_B = 0 \rightarrow$$

$$R_A \times 6 - 20 \times 4 \times 2 - \frac{1}{2} \times 4 \times 40 \times \frac{4}{3} = 0$$

$$\therefore \quad \mathbf{R_A = 44.44\ kN} \qquad \textbf{Ans.}$$

$$\sum V = 0 \rightarrow$$

$$R_A + R_B - 20 \times 4 - \frac{1}{2} \times 4 \times 40 = 0$$

∴ $R_B = 80 + 80 - 44.44,$ since $R_A = 44.44$ kN

∴ $\mathbf{R_B = 115.56}$ **kN** **Ans.**

Example 9.11 The beam *AB* of span 12 m shown in Fig. 9.29 is hinged at *A* and is on roller at *B*. Determine the reactions developed at *A* and *B* for due to the loading shown in the figure.

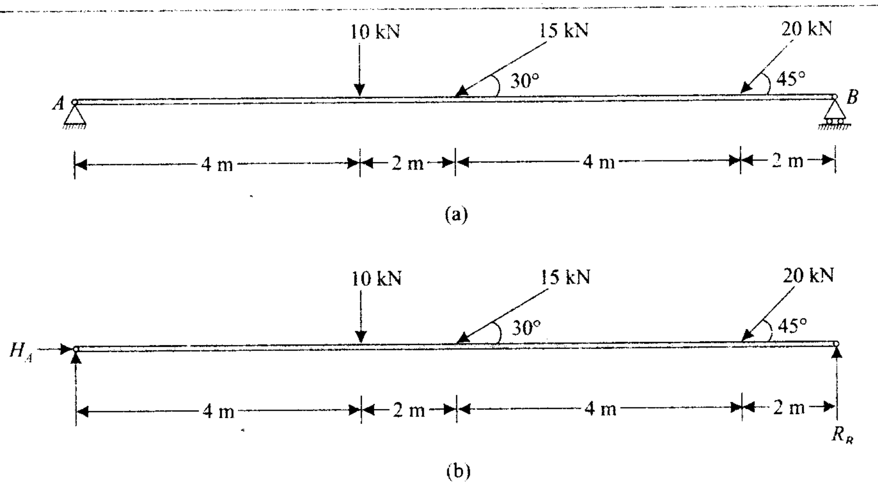


Fig. 9.29

Solution. At *A* the reaction can be in any direction. Let it be represented by its components V_A and H_A as shown in Fig. 9.29(b). At *B* the reaction is in vertical direction only. Let its magnitude be R_B. The beam is in equilibrium under the action of system of forces shown in Fig. 9.29(b).

Now,

$$\sum H = 0 \rightarrow$$

$$H_A - 15\cos 30 - 20\cos 45 = 0$$

∴ $\mathbf{H_A = 27.13}$ **kN** **Ans.**

$$\sum M_A = 0 \rightarrow$$

$$-R_B \times 12 + 10 \times 4 + 15\sin 30 \times 6 + 20\sin 45 \times 10 = 0$$

∴ $\mathbf{R_B = 18.87}$ **kN** **Ans.**

$$\sum V = 0 \rightarrow$$

$$V_A + R_B - 10 - 15\sin 30 - 20\sin 45 = 0$$

$$V_A + 18.87 - 10 - 15\sin 30 - 20\sin 45 = 0$$

$\mathbf{V_A = 12.770}$ **kN** **Ans.**

Example 9.12 Find the magnitude and direction of reactions at supports A and B in the beam AB shown in Fig. 9.30.

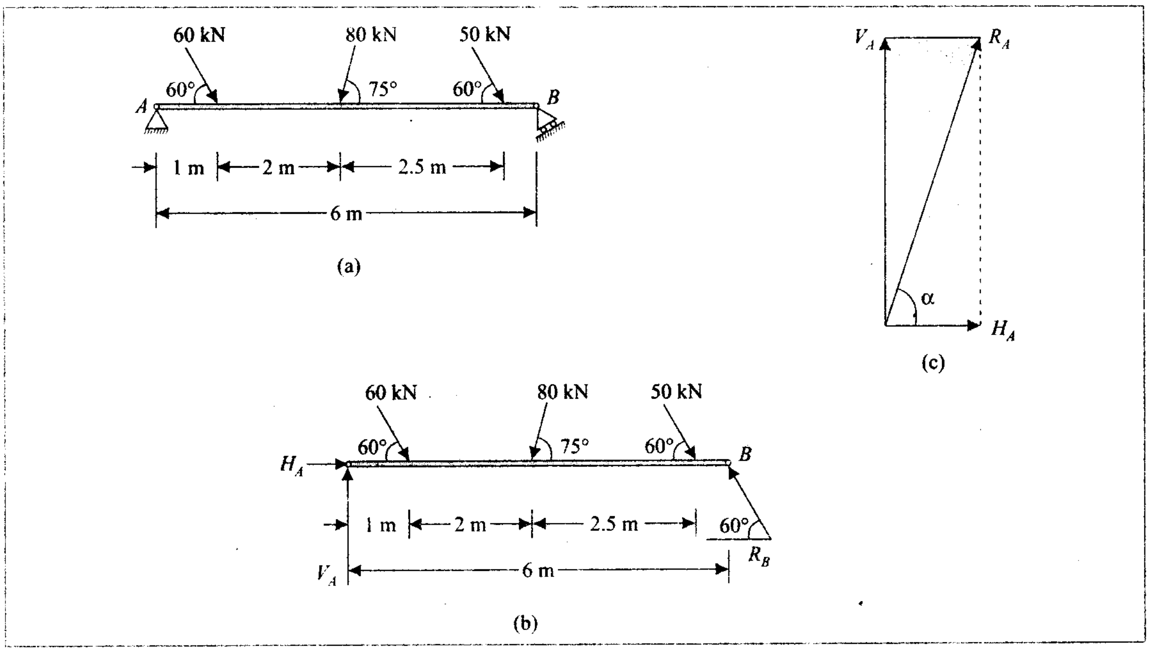


Fig. 9.30

Solution. The reaction R_B will be at right angles to the inclined support, i.e. at 90 - 30 = 60° to horizontal as shown in the Fig. 9.30(b). Let the components of reactions at A be V_A and H_A. Then

$$\sum M_A = 0 \rightarrow$$

$$60 \sin 60 \times 1 + 80 \sin 75 \times 3 + 50 \sin 60 \times 5.5 - R_B \sin 60 \times 6 = 0$$

∴ **R_B = 100.45 kN, at 60° to horizontal as shown in figure.** **Ans.**

$$\sum H = 0 \rightarrow$$

$$H_A + 60 \cos 60 - 80 \cos 75 + 50 \cos 60 - R_B \cos 60 = 0$$

$$H_A = - 60 \cos 60 + 80 \cos 75 - 50 \cos 60 + 100.45 \cos 60$$

∴ $H_A = 15.93$ kN

$$\sum V = 0 \rightarrow$$

$$V_A + R_B \sin 60 - 60 \sin 60 - 80 \sin 75 - 50 \sin 60 = 0$$

∴ $V_A = - 100.45 \sin 60 + 60 \sin 60 + 80 \sin 75 + 50 \sin 60$

i.e. $V_A = 85.54$ kN

∴ $R_A = \sqrt{H_A^2 + V_A^2}$

$= \sqrt{15.93^2 + 85.54^2}$

i.e., $\mathbf{R_A = 87.02\ kN}$ **Ans.**

$\alpha_A = \tan^{-1}\dfrac{85.54}{15.93}$ **= 79.45°, as shown in Fig. 9.30(c).** **Ans.**

Example 9.13 Find the reactions developed at supports A and B of the loaded beam shown in Fig. 9.31.

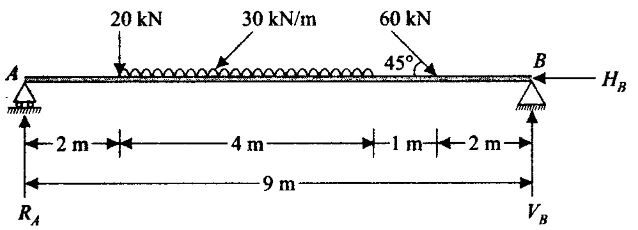


Fig. 9.31

Solution. Reaction at A is vertical, since roller support is horizontal. Let H_B and V_B be the components of the reaction at B.

$\sum M_B = 0 \rightarrow$

$R_A \times 9 - 20 \times 7 - 30 \times 4 \times 5 - 60 \sin 45 \times 2 = 0$

∴ $\mathbf{R_A = 91.65\ kN}$ **Ans.**

$\sum H_A = 0 \rightarrow$

$60 \cos 45 - H_B = 0$

∴ $\mathbf{H_B = 42.43\ kN}$ **Ans.**

$\sum V = 0$

$R_A + V_B - 20 - 30 \times 4 - 60 \sin 45 = 0$

∴ $V_B = 20 + 30 \times 4 + 60 \sin 45 - 91.65$, since $R_B = 91.65$ kN

i.e., $\mathbf{V_B = 90.78\ kN}$ **Ans.**

Example 9.14 Determine the reactions at supports A and B of the overhanging beam shown in Fig. 9.32.

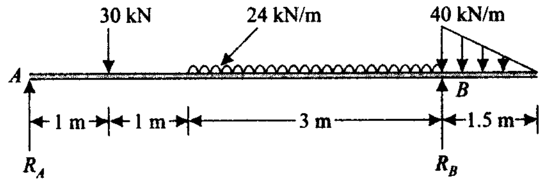


Fig. 9.32

Solution.

$$\sum M_A = 0 \rightarrow$$

$$30 \times 1 + 24 \times 3 \times (2 + 1.5) + \frac{1}{2} \times 1.5 \times 40 \times \left(5 + \frac{1}{3} \times 1.5\right) - R_B \times 5 = 0.$$

$\therefore$ $$R_B = 89.4 \text{ kN}$$ **Ans.**

$$\sum V = 0 \rightarrow$$

$$R_A + R_B - 30 - 24 \times 3 - \frac{1}{2} \times 1.5 \times 40 = 0.$$

$\therefore$ $$R_A = 30 + 72 + 30 - 85.4, \quad \text{since } R_B = 85.4 \text{ kN}$$

$$R_A = 42.6 \text{ kN}$$ **Ans.**

Example 9.15 Determine the reactions developed at supports A and B of overhanging beam shown in Fig. 9.33.

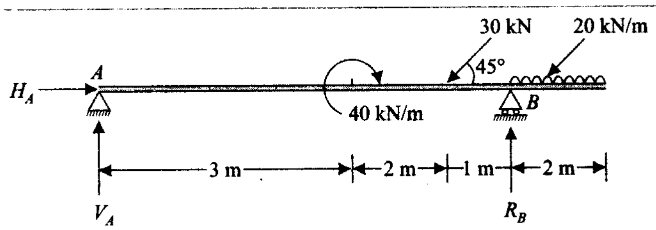


Fig. 9.33

Solution. Let the reaction components developed at support A be V_A and H_A and that at B be R_B

$$\sum M_A = 0 \rightarrow$$

$$40 + 30 \sin 45 \times 5 + 20 \times 2 \times 7 - R_B \times 6 = 0$$

$\therefore$ $$R_B = 71.01 \text{ kN}$$ **Ans.**

$\sum F_H = 0 \rightarrow$

$$H_A - 30\cos 45 \quad \therefore H_A = \mathbf{21.21\ kN}$$ **Ans.**

$\sum F_V = 0 \rightarrow$

$$V_A - 30\sin 45 + R_B - 20 \times 2 = 0.$$

$\therefore$ $$V_A = 30\sin 45 - 71.01 + 40, \quad \text{since } R_B = 71.01$$

i.e., $$V_A = -9.79 \text{ kN}$$

$$V_A = \mathbf{9.79 \downarrow kN}$$ **Ans.**

Example 9.16 Find the reactions developed at supports A and B of the loaded beam shown in Fig. 9.34.

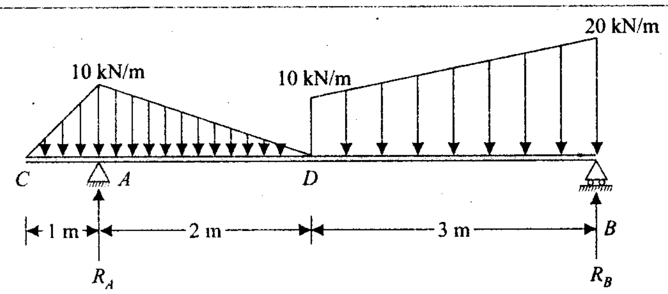


Fig. 9.34

Solution. The load is divided into a triangular load between C and A, another triangular load between A and D, a uniformly distributed load of 10 kN/m over portion DB and a triangular load of intensity zero at D and 10 kN [20 - 10 = 10 kN] at B.

$\sum M_B = 0 \rightarrow$

$$-\frac{1}{2} \times 1 \times 10 \times \left(\frac{1}{3} + 5\right) - \frac{1}{2} \times 2 \times 10 \times \left(5 - \frac{2}{3}\right) - 10 \times 3 \times 1.5 - \frac{1}{2} \times 3 \times 10 \times 1 + R_A \times 5 = 0.$$

$\therefore$ $$R_A = 26 \text{ kN}$$

$\sum F_V = 0 \rightarrow$

$$R_A + R_B - \frac{1}{2} \times 1 \times 10 - \frac{1}{2} \times 2 \times 10 - 10 \times 3 - \frac{1}{2} \times 3 \times 10 = 0$$

$$26 + R_B - 5 - 10 - 30 - 15 = 0$$

or $$R_B = \mathbf{34\ kN}$$ **Ans.**

Example 9.17 A beam 20 m long supported on two intermediate supports, 12 m apart, carries a udl of 6 kN/m and two concentrated loads of 30 kN at left end *A* and 50 kN at the right end *B* as shown in Fig. 9.35. How far away should the first support *C* be located from the end *A* so that the reactions at both the supports are equal?

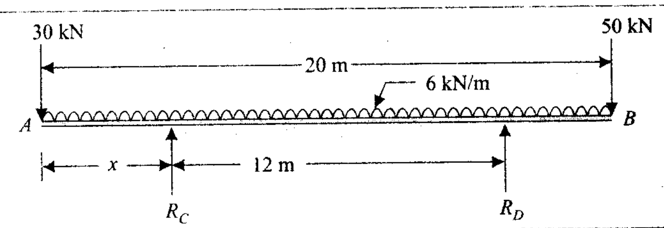


Fig. 9.35

Solution. Let the support *C* be at a distance *x* metres from end *A*.

Now, it is given that $R_C = R_D$

$\sum F_V = 0 \rightarrow$

$$R_C + R_D - 30 - 6 \times 20 - 50 = 0$$

i.e., $$2R_C = 30 + 120 + 50, \quad \text{since } R_D = R_C$$

$\therefore$ $$R_C = 100 \text{ kN}$$

Hence $$R_D = 100 \text{ kN}$$

$\sum M_A = 0$, gives

$$R_C x + R_D (12 + x) - 6 \times 20 \times 10 - 50 \times 20 = 0$$

$$100x + 100 (12 + x) - 1200 - 1000 = 0, \quad \text{since } R_C = R_D = 100 \text{ kN}$$

$\therefore$ $$x = 5 \text{ m}$$ **Ans.**

Example 9.18 A beam *AB* supports uniformly distributed load of intensity W_1 and rests on soil which exerts a uniformly varying upward reaction as shown in Fig. 9.36. Determine W_2 and W_3 corresponding to equilibrium. Note that soil can exert only an upward reaction at any point on the beam. Hence, state for what range of values of $\frac{a}{L}$ the results obtained are valid.

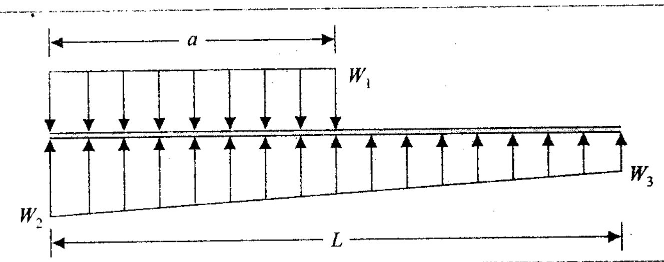


Fig. 9.36

Solution.

$$\sum F_V = 0 \rightarrow \qquad aW_1 = \frac{W_2 + W_3}{2} L \qquad \text{...(i)}$$

$$\sum M_A = 0 \rightarrow \qquad aW_1 \frac{a}{2} = W_3 L \times \frac{L}{2} + \frac{1}{2}(W_2 - W_3) L \times \frac{L}{3}$$

i.e.,
$$W_1 a^2 = W_3 L^2 + (W_2 - W_3)\frac{L^2}{3} \qquad \text{...(ii)}$$

From (i),
$$W_2 = \frac{2\,aW_1}{L} - W_3 \qquad \text{...(iii)}$$

From (ii) and (iii), we get

$$W_1 a^2 = W_3 L^2 + \left[\frac{2\,aW_1}{L} - W_3 - W_3\right]\frac{L^2}{3}$$

$$= \frac{2\,aL}{3} W_1 + W_3\left(1 - \frac{2}{3}\right) L^2$$

$\therefore$
$$W_1\left(a^2 - \frac{2\,aL}{3}\right) = W_3 \frac{L^2}{3}$$

$\therefore$
$$W_3 = \frac{3\,W_1}{L^2}\left(a^2 - \frac{2\,aL}{3}\right) \qquad \textbf{Ans.}$$

Substituting this value of W_3 in (iii) we get,

$$W_2 = \frac{2\,aW_1}{L} - \frac{3\,W_1}{L^2}\left(a^2 - \frac{2\,aL}{3}\right)$$

$\therefore$
$$W_2 = \left(\frac{4\,a}{L} - \frac{3\,a^2}{L^2}\right) W_1 \qquad \textbf{Ans.}$$

For the above expressions to be valid W_2 and W_3 should not be negative. The limiting cases are when they are zero.

$\therefore$
$$0 = \frac{3\,W_1}{L^2}\left(a^2 - \frac{2\,aL}{3}\right)$$

i.e.,
$$a = \frac{2\,L}{3} \quad \text{or} \quad \frac{a}{L} = \frac{2}{3}$$

and $$0 = \frac{4a}{L} - \frac{3a^2}{L^2}$$

i.e., $$\frac{a}{L} = \frac{4}{3}$$

Hence the results obtained are valid for the range $\frac{a}{L} = \frac{2}{3}$ to $\frac{4}{3}$ only. However, there is no possibility of loading beyond $\frac{a}{L} = 1$. Hence the **valid range of $\frac{a}{L}$ is from $\frac{2}{3}$ to 1.** **Ans.**

Example 9.19 Determine the reactions at *A, B* and *D* of the compound beam shown in Fig. 9.37(a). Neglect the self weight of the members.

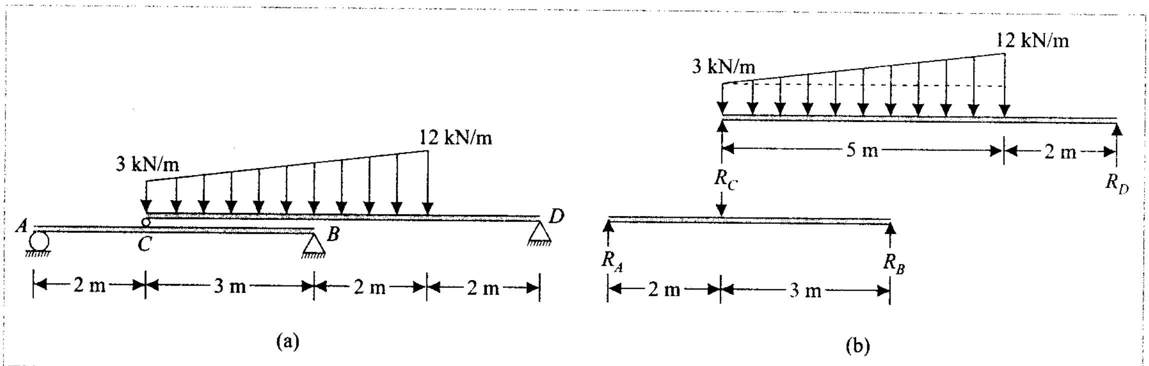


Fig. 9.37

Solution. Free body diagrams of beams *CD* and *AB* are as shown in Fig. 9.37(b). The load may be divided into a udl of 3 kN/m and a triangular load of intensity zero at *C* and 9 kN/m at 2 m from *D*.

Consider beam *CD*:

$\sum M_C = 0 \rightarrow$

$$-R_D \times 7 + 3 \times 5 \times 2.5 + \frac{1}{2} \times 5 \times 9 \times \frac{2}{3} \times 5 = 0$$

$\therefore$ $$\mathbf{R_D = 16.07\ kN}$$ **Ans.**

$\sum F_V = 0 \rightarrow$

$$R_C + R_D - 3 \times 5 - \frac{1}{2} \times 5 \times 9 = 0$$

$$R_C = 15 + 22.5 - 16.07, \quad \text{since } R_D = 16.07 \text{ kN}$$

i.e., $$R_C = 21.43 \text{ kN}$$

Consider beam AB:

$\sum M_A = 0 \rightarrow$

$$R_C \times 2 - R_B \times 5 = 0$$

$\therefore$ $$R_B = \frac{2 \times 21.43}{5}, \quad \text{since } R_C = 21.43 \text{ kN}$$

$\therefore$ $$\mathbf{R_B = 8.57 \text{ kN}}$$ **Ans.**

$\sum F_V = 0 \rightarrow$

$$R_A + R_B - R_C = 0$$

$$R_A + 8.57 - 21.43 = 0$$

$\therefore$ $$\mathbf{R_A = 12.86 \text{ kN}}$$ **Ans.**

Example 9.20 The beams AB and CF are arranged as shown in Fig. 9.38(a). Determine the reactions at A, C and D due to the loads acting on the beam as shown in the figure.

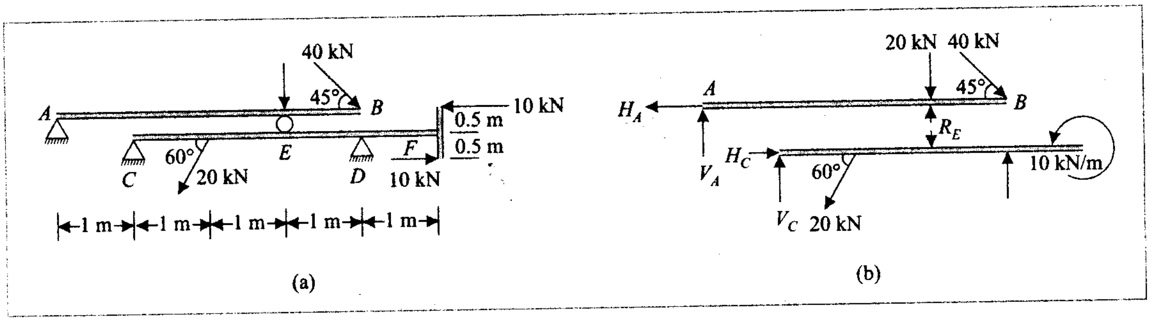


Fig. 9.38

Solution. The free body diagrams of beams AB and CF are as shown in Fig. 9.38(b). The two 10 kN forces constitute a couple moment of 10 × 1 = 10 kN-m as shown in Fig. 9.38(b).

Now consider beam AB:

$\sum M_A = 0 \rightarrow$

$$-R_E \times 3 + 20 \times 3 + 40 \sin 45 \times 4 = 0$$

$\therefore$ $$R_E = 57.71 \text{ kN}$$

$\sum F_H = 0 \rightarrow$

$$-H_A + 40 \cos 45 = 0$$

or $$\mathbf{H_A = 28.28 \text{ kN}}$$ **Ans.**

$\sum F_V = 0 \rightarrow$

$$V_A - 20 - 40 \sin 45 + R_E = 0$$

$$V_A = 20 + 40 \sin 45 - 57.71, \quad \text{since } R_E = 57.71 \text{ kN}$$

$\therefore$ $$V_A = -9.43 \text{ kN}$$

i.e., $$\boldsymbol{V_A = 9.43 \text{ kN, downward}}$$ **Ans.**

Consider beam *CF*:

$\sum M_C = 0 \rightarrow$

$$-R_D \times 3 + 20 \sin 60 \times 1 + R_E \times 2 - 10 = 0$$

$\therefore$ $$R_D = 20 \sin 60 - 10 + 57.71 \times 2, \quad \text{since } R_E = 57.71 \text{ kN}$$

$\therefore$ $$\boldsymbol{R_D = 40.91 \text{ kN}}$$ **Ans.**

$\sum F_H = 0 \rightarrow$

$$H_C - 20 \cos 60 = 0 \quad \therefore \boldsymbol{H_C = 10 \text{ kN}}$$ **Ans.**

$\sum F_V = 0 \rightarrow$

$$V_C - 20 \sin 60 - R_E + R_D = 0$$

$$V_C - 20 \sin 60 - 57.71 + 40.91 = 0$$

$\therefore$ $$\boldsymbol{V_C = 34.12 \text{ kN}}$$ **Ans.**

Important Definitions

1. Varignon's theorem states that the sum of moments of a system of coplanar forces about a moment centre is equal to the moment of their resultant about the same moment centre.
2. Statically determinate beams may be defined as beams in which all reaction components can be found using the equation of equilibrium only.

Important Formulae

1. Any one set of the following equations of equilibrium may be used for the analysis of non-concurrent coplanar system of forces.

   (i) $\Sigma F_x = 0, \quad \Sigma F_y = 0, \quad \Sigma M_A = 0$

   (ii) If *AB* is not in *x*-direction,

   $\Sigma F_x = 0, \quad \Sigma M_A = 0, \quad \Sigma M_B = 0$

   (iii) If *AB* is not in y-direction,

$\Sigma F_y = 0, \quad \Sigma M_A = 0, \quad \Sigma M_B = 0$

(iv) If A, B and C are not collinear,

$\Sigma M_A = 0, \quad \Sigma M_B = 0, \quad \Sigma M_C = 0$

Problems for Exercise

1. The frame shown in Fig. 9.39 is supported by a hinge at A and a roller at E. Compute the horizontal and vertical components of the reactions at hinges B and C as they act upon member AC.

[**Ans.** H_B = 200 N (towards right), V_B = 600 N, downwards]

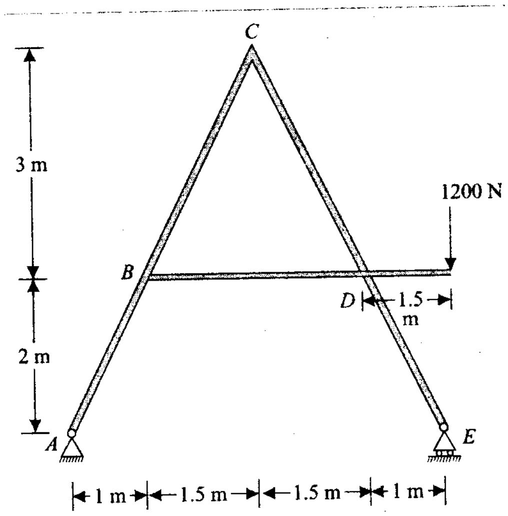


Fig. 9.39

2. The frame shown in Fig. 9.40 is supported by a hinge at E and by a roller at D. Determine the horizontal and vertical components of the reaction at hinge C as it acts upon member BD.

[**Ans.** H_C = 140 N, towards right, V_C = 35 N, upwards]

3. Beam AB, 8.5 m long, is hinged at A and has a roller support at B. The roller support is inclined at 45° to the horizontal. Find the reactions at A and B, if the loads acting are as shown in Fig. 9.41.

[**Ans.** H_A = 51.67 kN, V_A = 73.04 kN, R_B = 101.25 kN]

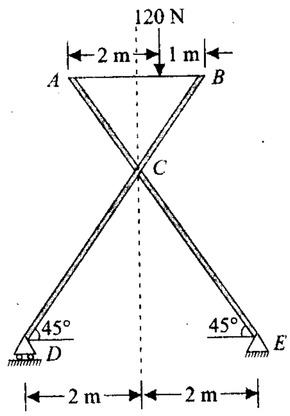


Fig. 9.40

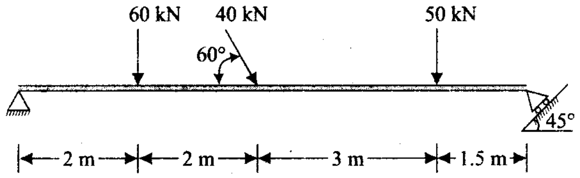


Fig. 9.41

4. Determine reactions developed at supports in the beam shown in Fig. 9.42.

[**Ans.** H_A = 21.21 kN, V_A = 9.80 kN, downward, R_B = 71.01 kN]

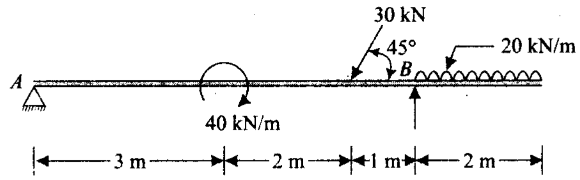


Fig. 9.42

5. Overhanging beam shown in Fig. 9.43 is on roller at A and is hinged at the end B. Determine the reactions developed for the loadings shown in figure.

[**Ans.** R_A = 160 kN, R_B = 90 kN]

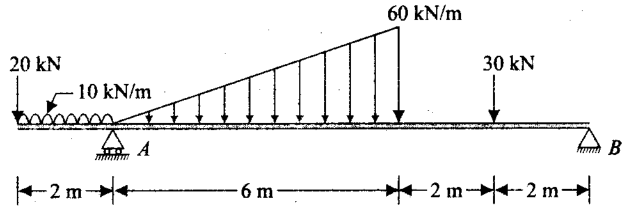


Fig. 9.43

6. Determine the reactions at supports A, C and D in the compound beam shown in Fig. 9.44.

[**Ans.** R_A = 20 kN, R_C = 67.5 kN, R_D = 7.5 kN, downward]

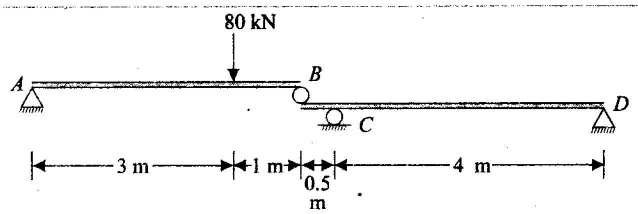


Fig. 9.44

CHAPTER 10

Friction

When a body moves or tends to move over another body, a force opposing the motion develops at the contact surfaces. The force which opposes the movement or the tendency of movement is called the **frictional force** or simply **friction**. So far, in earlier chapters, we had ignored this force and considered contacting surfaces are smooth. Actually in almost all cases the contacting surfaces are not smooth. There are minutely projecting particles which develop frictional force to oppose the tendency to movement of one surface over the other surface. In this chapter, the additional terminology used in connection with frictional forces are explained and laws of dry friction (wet friction excluded) are presented. Applications of these laws to many engineering problems are illustrated.

10.1 FRICTIONAL FORCE

Whenever a resultant force acts in the direction of contacting surfaces frictional force develops to oppose that force. The frictional force, like any other reaction, has a remarkable property of adjusting itself in magnitude to the tangential force. However, there is a limit beyond which the magnitude of the frictional force will not develop. If the applied tangential force is more than this maximum frictional force, there will be movement of one body over the other body with an acceleration as per Newton's second law of mass times acceleration equal to the resultant force. This maximum value of frictional force, which comes into play when the motion is impending is known as **Limiting Friction**. It may be noted that when the applied tangential force is less than the limiting friction, the body remains at rest and such friction is called **static friction**, which will have any value between zero and limiting friction. If the value of applied tangential force exceeds the limiting friction, the body starts moving over another body and the frictional resistance experienced while moving is known as **Dynamic Friction**. The magnitude of dynamic friction is found to be less than limiting friction. Dynamic friction may be further classified into two groups:

(i) *Sliding Friction:* It is the friction experienced by a body when it slides over the other body.

(ii) *Rolling Friction:* It is the friction experienced by a body when it rolls over another body.

It has been experimentally proved that, between two contacting surfaces, the magnitude of limiting friction bears a constant ratio to the normal reaction between the two and this ratio is called 'Coefficient of Friction'. Referring to Fig. 10.1 a body weighing W is being pulled by a force P and the motion is impending. Let N be normal reaction and F the limiting frictional force. Then

$$\text{Coefficient of Friction} = \frac{F}{N}$$

Coefficient of friction is denoted by μ. Then

$$\mu = \frac{F}{N} \qquad \text{Eqn. (10.1)}$$

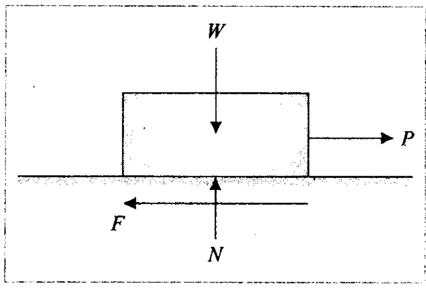


Fig. 10.1

10.2 LAWS OF FRICTION

The principles discussed in a previous chapter are mainly due to the experiemental studies by Coulomb (1781) and by Mozin (1831). These principles constitute the laws of dry friction and may be called as *Coulomb's laws of dry friction*. These laws are listed below:

(i) The frictional force always acts in a direction opposite to that in which the body tends to move.

(ii) Till the limiting value is reached, the magnitude of frictional force is exactly equal to the tangential force which tends to move the body.

(iii) The magnitude of the limiting friction bears a constant ratio to the normal reaction between the two contacting surfaces.

(iv) The force of friction depends upon the roughness/smoothness of the surfaces.

(v) The force of friction is independent of the area of contact between the two surfaces.

(vi) After the body starts moving, the dynamic friction comes into play, the magnitude of which is less than that of limiting friction and it bears a constant ratio to the normal force. This ratio is called coefficient of dynamic friction.

10.3 ANGLE OF FRICTION, ANGLE OF REPOSE AND CONE OF FRICTION

Angle of Friction

Consider the blowk shown in Fig. 10.2 subject to pull P. Let F be the frictional force developed and N the normal reaction. Thus, at contact surface, the reactions are F and N. They can be combined to get the resultant reaction R which acts at angle θ to normal reaction. This angle is given by

$$\tan\theta = \frac{F}{N}$$

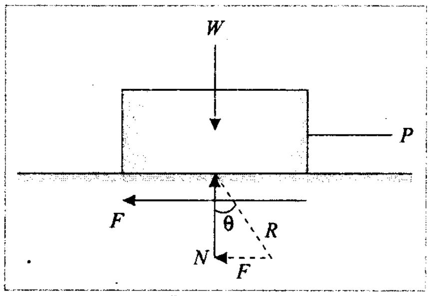


Fig. 10.2

As frictional force increases the angle θ increases and it can reach maximum value α when limiting value of friction is reached. Thus, when motion is impending

$$\tan \alpha = \frac{F}{N} = \mu \qquad \text{Eqn. (10.2)}$$

and this value of α is called *angle of limiting friction*. Hence, the angle of limiting friction can be defined as the angle between the resultant reaction and the normal to the plane on which the motion of the body is impending.

Angle of Repose

It is very well-known that when grains (food grains, sand, cement, soil etc.) are heaped, there exists a limit for the inclination of the heap. Beyond that the grains start rolling down. The limiting angle upto which the grains repose (sleep) is called *angle of repose.*

Now consider the block of weight W shown in Fig. 10.3 which is resting on an inclined plane that makes angle θ with the horizontal, when θ is a small block, that rests on the plane. If θ is increased gradually a stage is reached at which the block will start sliding. The angle made by the plane with the horizontal is called angle of friction for the contacting surfaces. Thus, the maximum inclination of the plane on which the body, free from external forces, can repose is called *angle of repose.*

Consider the equilibrium of the block shown in Fig. 10.3. Since the surface of contact is not smooth, not only normal reaction but frictional force also develops. As the body tends to slide down, the frictional resistance will be up the plane.

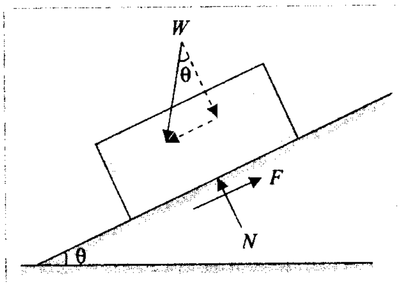


Fig. 10.3

Σ Forces normal to plane = 0, gives

$$N = W \cos \theta \qquad \text{...(i)}$$

Σ Forces parallel to plane = 0, gives

$$F = W \sin \theta \qquad \text{...(ii)}$$

Dividing eqn. (ii) by eqn. (i) we get,

$$\frac{F}{N} = \tan \theta$$

If ϕ is the value of θ when motion is impending, frictional force will be limiting friction and hence

$$\tan \phi = \frac{F}{N} \qquad \text{Eqn. (10.3)}$$

i.e.,

$$\tan \phi = \mu = \tan \alpha$$

or

$$\phi = \alpha.$$

Thus, the value of angle of repose is the same as the value of limiting angle of friction.

Cone of Friction

When a body is having impending motion in the direction of P, the frictional force will be the limiting friction and the resultant reaction R will make limiting frictional angle α with the normal as shown in Fig. 10.4. If the body is having impending motion in some other direction, the resultant reaction makes limiting frictional angle α with the normal. Thus, if the direction of force P is gradually changed through 360°, the resultant R generates a right circular cone with semicentral angle equal to α.

If the resultant reaction lies on the surface of this inverted right circular cone whose semicentral angle is limiting frictional angle α, the motion of the body is impending. If the resultant is within this cone the body is stationary. This inverted cone with semi-central angle, equal to limiting frictional angle α, is called *cone of friction*.

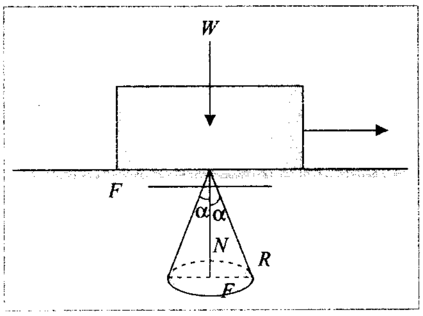


Fig. 10.4

10.4 PROBLEMS ON BLOCKS RESTING ON HORIZONTAL AND INCLINED PLANES

Analysis of such problems, when motion is impending is illustrated in this article by solving typical problems.

Example 10.1 Block A weighing 1000 N rests over block B which weighs 2000 N as shown in Fig. 10.5(a). Block A is tied to a wall with a horizontal string. If the coefficient of friction between A and B is 1/4 and that between B and the floor is 1/3, what value of force P is required to create impending motion if (a) P is horizontal, (b) P acts 30° upwards to horizontal?

Solution. (a) When P is horizontal: The free body diagrams of the two blocks are shown in Fig. 10.5(b). Note the frictional forces are to be marked in the opposite directions of impending relative motion. In this problem, block B is having impending motion to the right. Hence on it F_1

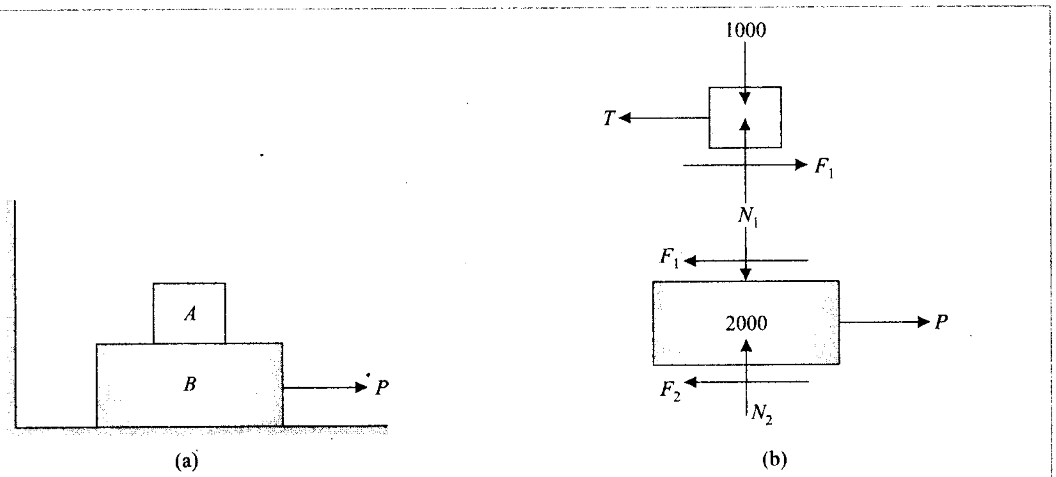


Fig. 10.5

and F_2 are towards right. The relative motion of block A w.r.t. B is to the left. Hence the direction of F_1 in this is towards the right. Another way of thinking for the direction of F_1 in case of block A can be 'actions and reactions are equal and opposite'. Hence on block B if F_1 is towards left, on A it should be towards right.

Now consider the equilibrium of block A.

$$\sum F_V = 0 \rightarrow$$

$$N_1 - 1000 = 0 \quad \text{or} \quad N_1 = 1000 \text{ newton.}$$

Since F_1 is limiting friction,

$$\frac{F_1}{N_1} = \frac{1}{4}$$

$$\therefore \qquad F_1 = \frac{1}{4} \times 1000 = 250 \text{ newton.}$$

$$\sum F_H = 0 \rightarrow$$

$$F_1 - T = 0 \quad \text{or} \quad T = F_1, \quad \text{i.e.} \quad T = 250 \text{ newton.}$$

Consider the equilibrium of block B.

$$\sum F_V = 0 \rightarrow$$

$$N_2 - N_1 - 2000 = 0.$$

$$\therefore \qquad N_2 = N_1 + 2000 = 1000 + 2000 = 3000 \text{ newton.}$$

Since F_2 is limiting friction,

$$F_2 = \mu_2 N_2 = \frac{1}{3} \times 3000 = 1000 \text{ newton.}$$

$$\sum F_H = 0 \rightarrow$$

$$P - F_1 - F_2 = 0$$

$$\therefore \qquad P = F_1 + F_2 = 250 + 1000 = \mathbf{1250\ newton.}$$ **Ans.**

(b) When P is inclined:

Free body diagrams for this case are shown in Fig. 10.5(c).

Considering equilibrium of block A, we get

$$\sum F_V = 0 \quad \rightarrow \quad N_1 = 1000 \text{ newton.}$$

$$\therefore \quad F_1 = \frac{1}{4} \times 1000 = 250 \text{ newton.}$$

$$\sum F_H = 0 \quad \rightarrow \quad T = F_1 = 250 \text{ newton.}$$

Consider the equilibrium of block B.

$$\sum F_V = 0 \rightarrow$$

$$N_2 - 2000 - N_1 + P \sin 30 = 0$$

or $$N_2 + 0.5P = 3000, \text{ since } N_1 = 1000 \text{ newton.}$$

From law of friction

$$F_2 = \mu_2 N_2 = \frac{1}{3} \times (3000 - 0.5P)$$

$$= 1000 - \frac{0.5}{3} P.$$

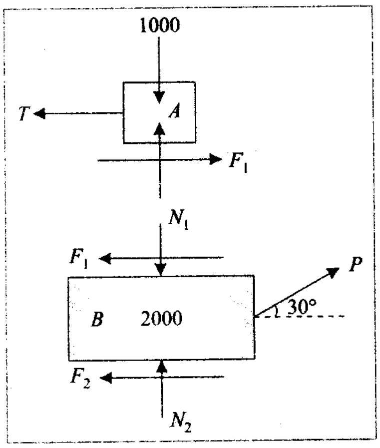


Fig. 10.5(c)

$$\sum F_H = 0 \rightarrow$$

$$P \cos 30 - F_1 - F_2 = 0$$

$$\therefore \quad P \cos 30 - 250 - \left(1000 - \frac{0.5}{3} P\right) = 0$$

$$\therefore \quad P\left(\cos 30 + \frac{0.5}{3}\right) = 1250$$

$$\therefore \quad \mathbf{P = 1210.4 \ newton}$$ **Ans.**

Example 10.2 What should be the value of θ in Fig. 10.6(a) which will make the motion of 900 N block down the plane to impend? The coefficient of friction for all contact surfaces is 1/3.

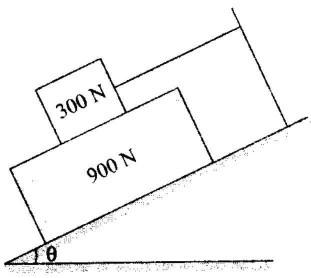


Fig. 10.6(a)

Solution. 900 N block is on the verge of moving downward. Hence frictional forces F_1 and F_2 [Ref. Fig. 10.6(b)] act up the plane on 900 N block. Free body diagrams of the blocks are as shown in Fig. 10.6(b).

Consider the equilibrium of 300 N block.

Σ Forces normal to plane = 0 →

$$N_1 - 300 \cos \theta = 0 \quad \text{or} \quad N_1 = 300 \cos \theta \qquad ...(i)$$

From law of friction,

$$F_1 = \frac{1}{3} N_1 = 100 \cos \theta \qquad \text{...(ii)}$$

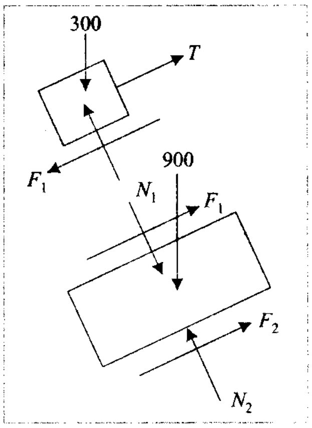


Fig. 10.6(b)

For 900 N block:

Σ Forces normal to plane = 0 $\rightarrow$

$$N_2 - N_1 - 900 \cos \theta = 0$$

or

$$N_2 = N_1 + 900 \cos \theta$$
$$= 300 \cos \theta + 900 \cos \theta$$
$$= 1200 \cos \theta.$$

From law of friction,

$$F_2 = \mu_2 N_2 = \frac{1}{3} \times 1200 \cos \theta = 400 \cos \theta.$$

Σ Forces parallel to the plane = 0 $\rightarrow$

$$F_1 + F_2 - 900 \sin \theta = 0$$

$$100 \cos \theta + 400 \cos \theta = 900 \sin \theta$$

$$\therefore \quad \tan \theta = \frac{500}{900}$$

$$\therefore \quad \theta = 29.05°$$ **Ans.**

Example 10.3 A block weighing 500 N just starts moving down a rough inclined plane when it is subjected to 200 N force acting up the inclined plane and it is at the point of moving up the plane when pulled up by a force of 300 N parallel to the plane. Find the inclination of the plane and the coefficient of friction between the inclined plane and the block.

Solution. Free body diagram of the block when its motion is impending down the plane is shown in Fig. 10.7(a) and that when it is moving up the plane is shown in Fig. 10.7(b).

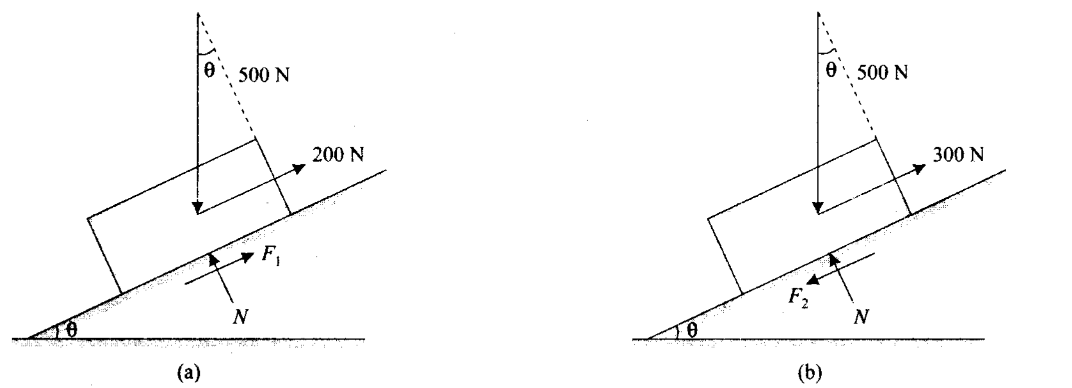


Fig. 10.7

When block starts moving down the plane [Ref. Fig. 5.7(a)]

Frictional forces oppose the direction of the movement. Hence F_1 is up the plane and F_2 down the plane. Since it is limiting case

$$\frac{F}{N} = \mu.$$

Σ forces perpendicular to the plane = 0 →

$$N - 500 \cos \theta = 0 \quad \text{or} \quad N = 500 \cos \theta \qquad \text{...(i)}$$

From law of friction,

$$F_1 = \mu N = 500 \mu \cos \theta \qquad \text{...(ii)}$$

Σ Forces parallel to the plane = 0 →

$$F_1 + 200 - 500 \sin \theta = 0$$

Substituting the value of F_1 from eqn. (ii), we get

$$500 \sin \theta - 500 \mu \cos \theta = 200 \qquad \text{...(iii)}$$

When the block starts moving up the plane [Fig. 10.7(b)]

Σ Forces perpendicular to the plane = 0 →

$$N - 500 \cos \theta = 0 \quad \text{i.e.,} \quad N = 500 \cos \theta \qquad \text{...(iv)}$$

From law of fraction, $F_2 = \mu N = 500 \mu \cos \theta$ ...(v)

Σ Forces parallel to the plane = 0 →

$$300 - 500 \sin \theta - F_2 = 0$$

i.e.,

$$500 \sin \theta + 500 \mu \cos \theta = 300 \qquad \text{...(vi)}$$

Adding eqns. (iii) and (vi), we get

$$1000 \sin \theta = 500$$

i.e.,

$$\sin \theta = 0.5$$

Hence $\theta = \mathbf{30°}$ **Ans.**

Substituting it in eqn. (vi), we get

$$500 \sin 30 + 500 \mu \cos 30 = 300$$

$$500 \mu \cos 30 = 300 - 250 = 50$$

$$\therefore \quad \mu = \frac{50}{500 \cos 30} = \mathbf{0.115} \qquad \textbf{Ans.}$$

Example 10.4 Block *A* weighing 1000 N and block *B* weighing 500 N are connected by flexible wire. The coefficient of friction between block *A* and the plane is 0.5 while that for block *B* and the plane is 0.2. Determine what value of inclination of the plane the system will have impending motion down the plane? [Ref. Fig. 10.8].

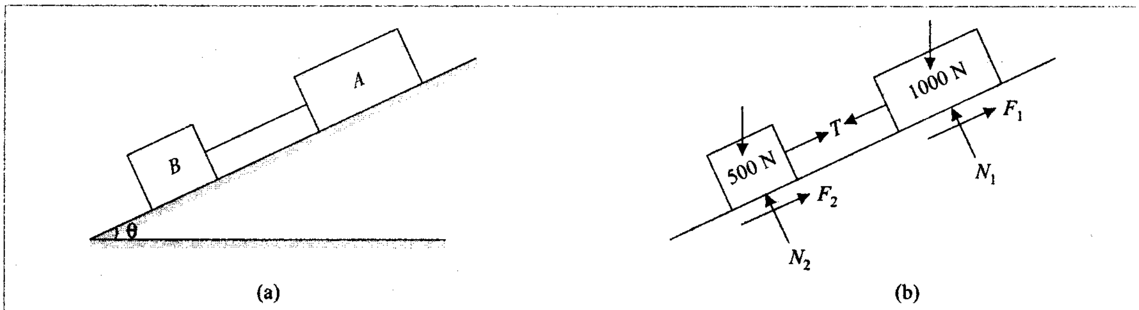


Fig. 10.8

Solution. Let θ be the inclination of the plane for which motion is impending. Free body diagrams of blocks *A* and *B* are as shown in Fig. 10.8(b). Considering equilibrium of block *A*,

Σ Forces normal to plane = 0 →

$$N_1 - 1000 \cos \theta = 0 \quad \text{or} \quad N_1 = 1000 \cos \theta \qquad \text{...(i)}$$

∴ From law of friction

$$F_1 = \mu_1 N_1 = 0.5 \times 1000 \cos \theta = 500 \cos \theta \qquad \text{...(ii)}$$

Σ Forces parallel to plane = 0 →

$$F_1 - T - 1000 \sin \theta = 0$$

or

$$T = 500 \cos \theta - 1000 \sin \theta \qquad \text{...(iii)}$$

Consider the equilibrium of block *B*,

Σ Forces normal to plane = 0 →

$$N_2 - 500 \cos \theta = 0 \quad \text{or} \quad N_2 = 500 \cos \theta \qquad \text{...(iv)}$$

From law of friction,

$$F_2 = \mu_2 N_2 = 0.2 \times 500 \cos \theta = 100 \cos \theta \qquad \text{...(v)}$$

Σ Forces parallel to plane = 0 →

$$F_2 + T - 500 \sin \theta = 0$$

Using the values of F_2 and T from eqn. (v) and eqn. (iii),

$$100 \cos \theta + 500 \cos \theta - 1000 \sin \theta - 500 \sin \theta = 0$$

$$600 \cos \theta = 1500 \sin \theta$$

∴
$$\tan \theta = \frac{600}{1500}$$

∴
$$\theta = \mathbf{21.8°}$$ **Ans.**

Example 10.5 What is the value of *P* in the system shown in Fig. 10.9(a) to cause the motion to impend? Assume the pulley is smooth and coefficient of friction between the other contact surfaces is 0.2.

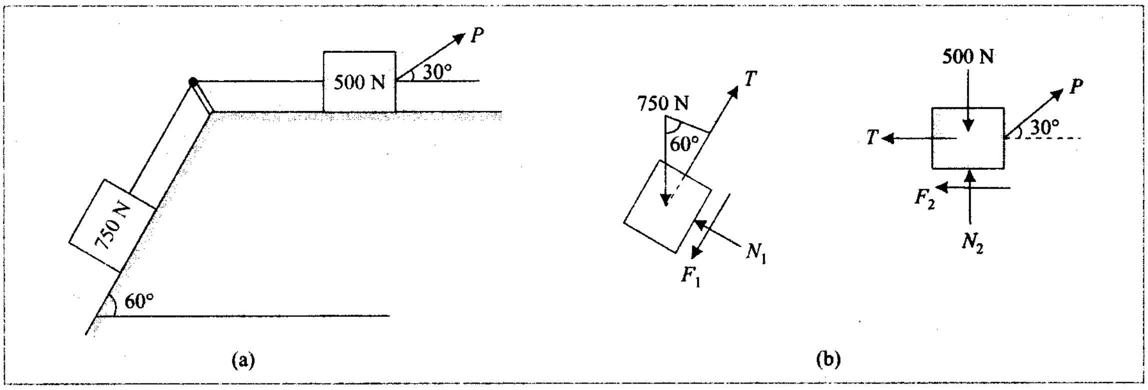


Fig. 10.9

Solution. Free body diagrams of the blocks are as shown in Fig. 10.9(b). Consider the equilibrium of 750 N block.

Σ Forces normal to the plane = 0 $\rightarrow$

$$N_1 - 750 \cos 60 = 0 \quad \therefore \; N_1 = 375 \text{ newton} \qquad \text{...(i)}$$

Since the motion is impending, from law of friction,

$$F_1 = \mu N_1 = 0.2 \times 375 = 75 \text{ newton} \qquad \text{...(ii)}$$

Σ Forces parallel to the plane = 0 $\rightarrow$

$$T - F_1 - 750 \sin 60 = 0$$

$\therefore$ $$T = 75 + 750 \sin 60 = 724.5 \text{ newton.} \qquad \text{...(iii)}$$

Consider the equilibrium of 500 N block.

$$\sum F_V = 0 \rightarrow$$

$$N_2 - 500 + P \sin 30 = 0$$

i.e., $$N_2 + 0.5P = 500 \qquad \text{...(iv)}$$

From law of friction,

$$F_2 = \mu N_2 = 0.2\,(500 - 0.5P) = 100 - 0.1P \qquad \text{...(v)}$$

$$\sum F_H = 0 \rightarrow$$

$$P \cos 30 - T - F_2 = 0$$

i.e., $$P \cos 30 - 724.5 - 100 + 0.1P = 0$$

$\therefore$ $$\mathbf{P = 853.5 \text{ N}}$$ **Ans.**

Example 10.6 Two identical planes AC and BC, inclined at 60° and 30° to the horizontal meet at C as shown in Fig. 10.10. A load of 1000 N rests on the inclined plane BC and is tied by a rope passing over a pulley to a block weighing W newtons and resting on the plane AC. If the coefficient of friction between the load and the plane BC is 0.28 and that between the block and the plane AC is 0.20, find the least and greatest values of W for the equilibrium of the system.

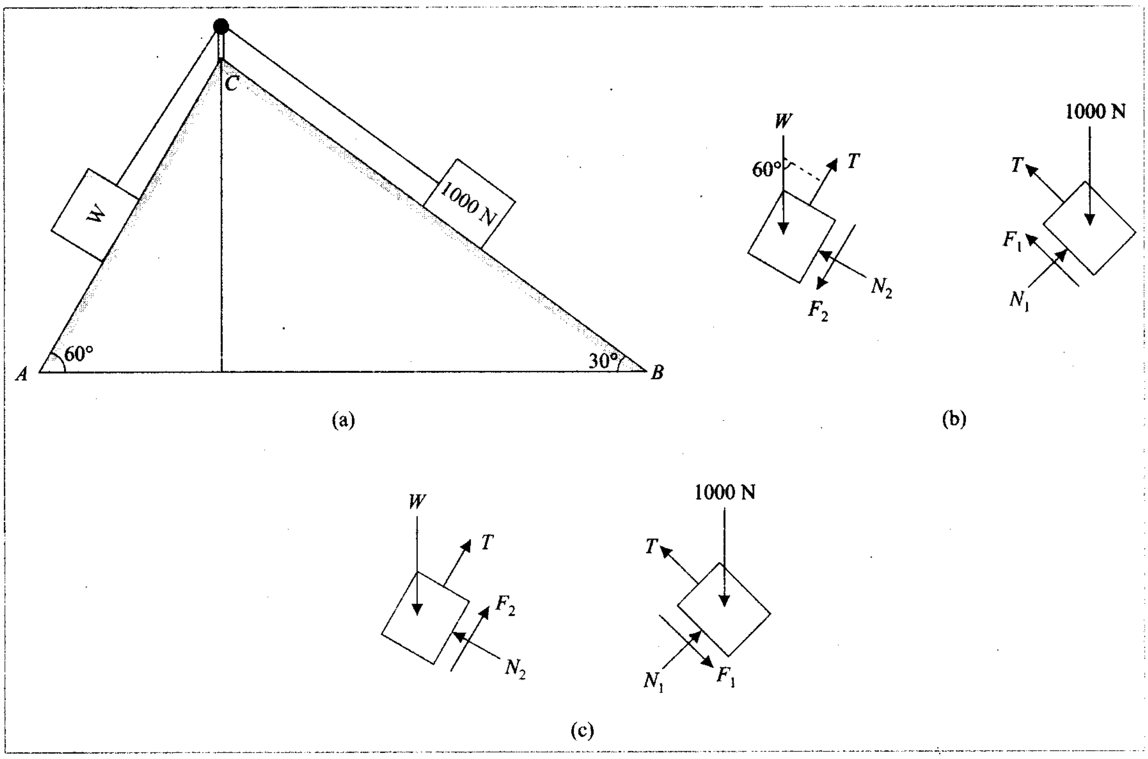


Fig. 10.10

Solution.

(a) Least value of W:

In this case motion of 1000 N block is impending down the plane and block W has impending motion up the plane. Hence free body diagrams for the blocks are as shown in Fig. 10.10(b). Considering the equilibrium of 1000 N block,

Σ Forces normal to the plane = 0 $\rightarrow$

$$N_1 - 1000 \cos 30 = 0 \quad \therefore \; N_1 = 866.0 \text{ newton} \qquad \text{...(i)}$$

From the law of friction

$$F_1 = \mu_1 N_1 = 0.28 \times 866.0 = 242.5 \text{ newton} \qquad \text{...(ii)}$$

Σ Forces parallel to the plane = 0 $\rightarrow$

$$T - 1000 \sin 30 + F_1 = 0$$

∴ $$T = 500 - 242.5 = 257.5 \text{ newton} \quad ...(iii)$$

Now consider the equilibrium of block weighing W.

Σ Forces normal to the plane = 0 →

$$N_2 - W\cos 60 = 0 \quad \text{i.e.,} \quad N_2 = 0.5\ W \quad ...(iv)$$

From law of friction

$$F_2 = \mu_2 N_2 = 0.2 \times 0.5\ W = 0.1\ W \quad ...(v)$$

Σ Forces parallel to the plane = 0 →

$$T - F_2 - W\sin 60 = 0$$

Substituting the values of T and F_2 from eqns. (iii) and (v), we get

$$257.5 - 0.1\ W - W\sin 60 = 0$$

∴ $$W = \frac{257.5}{0.1 + \sin 60} = \mathbf{266.6\ N.} \quad \text{...Ans.}$$

(b) For the greatest value of W:

In such case 1000 N block is on the verge of moving up the plane and W is on the verge of moving down the plane. For this case free body diagrams of the blocks are as shown in Fig. 10.10(c).

Considering the block of 1000 N,

Σ Forces normal to plane = 0 →

$$N_1 - 1000\cos 30 = 0 \quad \therefore N_1 = 866.0 \text{ newton} \quad ...(vi)$$

From law of friction,

$$F_1 = \mu_1 N_1 = 0.28 \times 866.0 = 242.5 \text{ N} \quad ...(vii)$$

Σ Forces parallel to the plane = 0 →

$$T - 1000\sin 30 - F_1 = 0$$

∴ $$T = 500 + 242.5 = 742.5 \text{ newton} \quad ...(viii)$$

Considering the equilibrium of block weighing W,

Σ Forces normal to plane = 0 →

$$N_2 - W\cos 60 = 0 \quad \text{or} \quad N_2 = 0.5\ W \quad ...(ix)$$

∴ $$F_2 = \mu_2 N_2 = 0.2 \times 0.5W = 0.1W \quad ...(x)$$

Σ Forces parallel to plane = 0 →

$$T - W\sin 60 + F_2 = 0 \quad ...(xi)$$

Substituting the values of T and F_2 from eqns. (viii) and (x), we get,

$$724.5 - W\sin 60 + 0.1\ W = 0$$

or $$W = \frac{742.5}{\sin 60 - 0.1} = \mathbf{969.3\ newton} \quad \textbf{Ans.}$$

The system of blocks are, in equilibrium for W = 266.6 N to 969.3 N.

Example 10.7 Two blocks connected by a horizontal link AB are supported on two rough planes as shown in Fig. 10.11(a). The coefficient of friction on the horizontal plane is 0.4. The limiting angle of friction for block B on the inclined plane is 20°. What is the smallest weight W of the block A for which equilibrium of the system can exist if weight of block B is 5 kN?

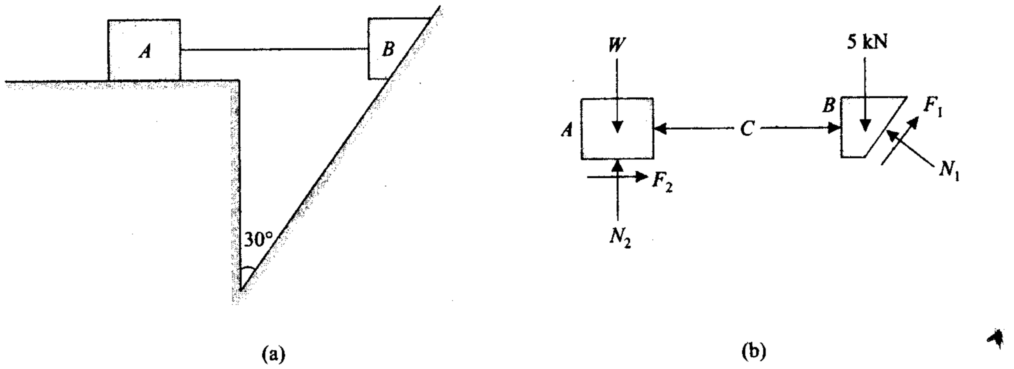


Fig. 10.11

Solution. Free body diagrams for blocks A and B are as shown in Fig. 10.11(b). Consider block B.

$$\mu = \tan 20°, \text{ given.}$$

$$\therefore \quad F_1 = N_1 \tan 20°$$

$$\sum F_V = 0 \rightarrow$$

$$N_1 \sin 30 + F_1 \sin 60 - 5 = 0$$

$$0.5N_1 + N_1 \tan 20 \sin 60 = 5$$

$$N_1 = 6.133 \text{ kN}$$

Hence,
$$F_1 = 6.13 \tan 20 = 2.232 \text{ kN}$$

$$\sum F_H = 0 \rightarrow$$

$$C + F_1 \cos 60 - N_1 \cos 30 = 0$$

$$C + 2.232 \cos 60 - 6.133 \cos 30 = 0$$

$$\therefore \quad C = 4.196 \text{ kN}$$

Now consider the equilibrium of block A.

$$\sum F_H = 0 \rightarrow$$

$$F_2 - C = 0 \quad \text{or} \quad F_2 = C = 4.196 \text{ kN}$$

From law of friction,

$$F_2 = \mu N_2$$

i.e.,
$$4.196 = 0.4\ N_2$$

∴
$$N_2 = 10.49 \text{ kN}$$

Then,
$$\sum F_V = 0 \rightarrow$$

$$N_2 - W = 0$$

or
$$W = N_2 = \mathbf{10.49\ kN}$$
Ans.

10.5 APPLICATION TO WEDGE PROBLEMS

Wedges are small pieces of hard materials with two of their opposite surfaces not parallel to each other. They are used to slightly lift heavy blocks, machinery, precast beams etc. for making final alignment or to make place for inserting lifting devices. In any problem weight of wedge is very small compared to the weight lifted. Hence in all problems self weight of wedge is neglected. It is found that in the analysis instead of treating normal reaction and frictional force independently, it is advantageous to treat their resultant.

If F is limiting friction, then resultant R makes limiting angle α with the normal. Its direction should be marked correctly. Note that the tangential component of the resultant reaction R is the frictional force and it will always oppose impending motion. Application to wedge problems is illustrated below by solving problems.

Example 10.8 Determine the force P required to start the movement of the wedge as shown in Fig. 10.12(a). The angle of friction for all surfaces of contact is 15°.

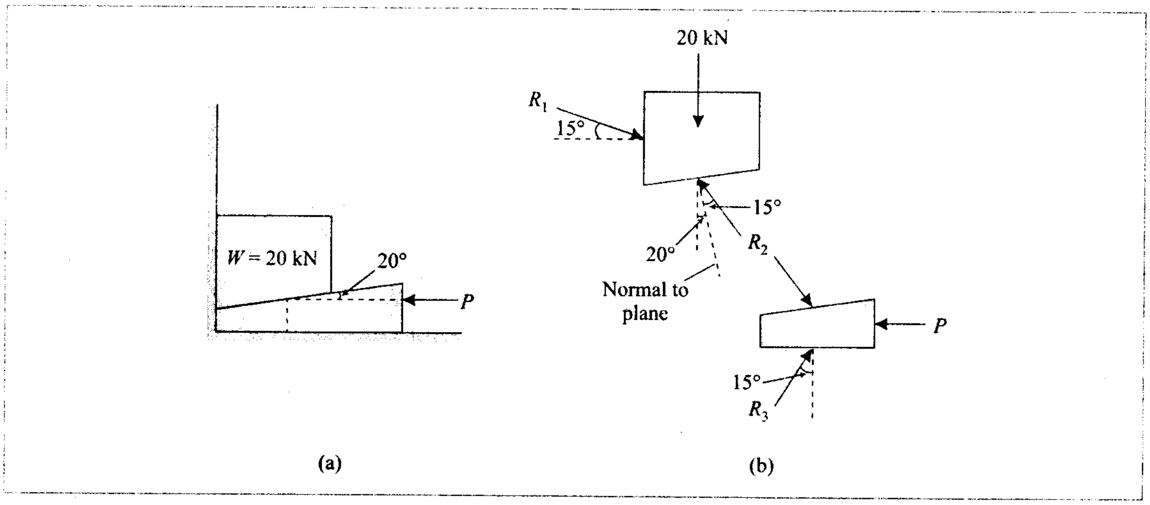


Fig. 10.12 (Contd.)

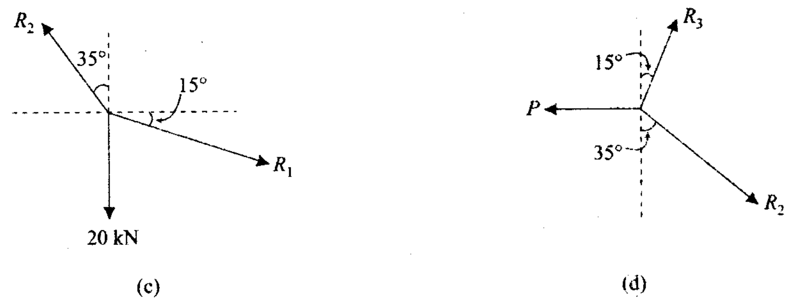


Fig. 10.12

Solution. As wedge is driven, it moves towards left and the block moves upwards. When motion is impending limiting friction develops. Hence resultant force makes limiting angle of 15° with normal. The care is taken to mark 15° inclination such that th tangential component of the resultant opposes the impending motion.

The free body diagrams of the block and wedge are shown in Fig. 10.12(b). The forces on block and wedge are redrawn in Figs. 10.12(c) and (d) so that Lami's theorem can be applied conveniently. Applying Lami's theorem to the system of forces on block

$$\frac{R_1}{\sin(180-15-20)} = \frac{R_2}{\sin(90-15)} = \frac{20}{\sin(15+20+90+15)}$$

i.e.,

$$\frac{R_1}{\sin 145} = \frac{R_2}{\sin 75} = \frac{20}{\sin 140}$$

∴ $$R_1 = 17.85 \text{ kN}$$

and $$R_2 = 30.05 \text{ kN}$$

Applying Lami's theorem to system of forces on the wedge, we get

$$\frac{P}{\sin 130} = \frac{R_2}{\sin 105}$$

∴ $$\mathbf{P = 23.84 \text{ kN}}$$ **Ans.**

Example 10.9 A block weighing 160 kN is to be raised by means of the wedges *A* and *B* as shown in Fig. 10.13(a). Find the value of force *P* for impending motion of block C upwards, if coefficient of friction is 0.25 for all contact surfaces. The self weight of wedges may be neglected.

Solution. Let ϕ be the angle of limiting friction.

∴ $$\phi = \tan^{-1}(0.25) = 14.036°$$

The free body diagrams of wedges *A*, *B* and block *C* are shown in Fig. 10.13(b). The problem being symmetric, the reactions R_1 and R_2 on wedges *A* and *B* are equal. The system of forces on block *C* and on wedge *A* are shown in the form convenient for applying Lami's theorem [Ref. Figs. 10.13(c) and (d)].

Consider the equilibrium of block C.

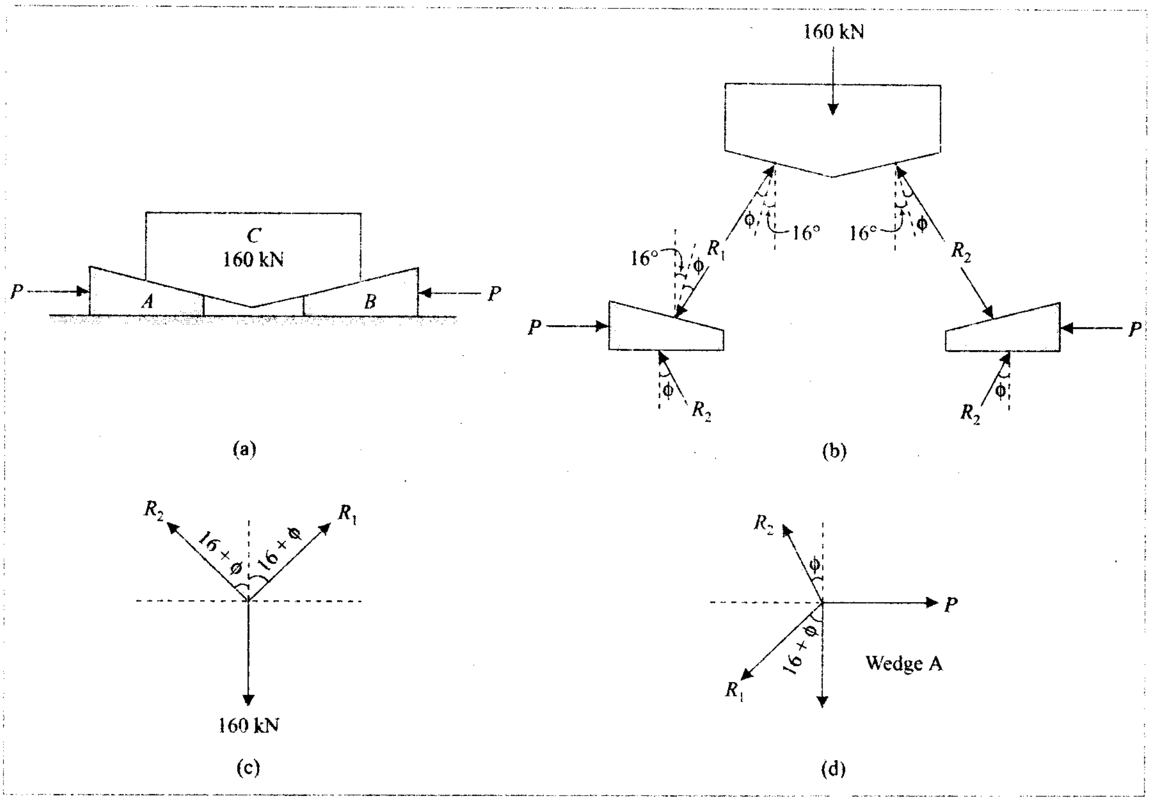


Fig. 10.13

$$\frac{R_1}{\sin(180-16-\phi)} = \frac{160}{\sin 2(\phi+16)}$$

i.e.,
$$\frac{R_1}{\sin 149.96} = \frac{160}{\sin 60.072}, \quad \text{since } \phi = 14.036°.$$

∴
$$R_1 = 92.42 \text{ kN}$$

Consider the equilibrium of the wedge A. Applying Lami's theorem, we get

$$\frac{P}{\sin(180-\phi-\phi-16)} = \frac{R_1}{\sin(90+\phi)}$$

∴
$$P = \frac{92.42 \sin 135.928}{\sin 104.036}, \quad \text{since } \phi = 14.036.$$

i.e.,
$$\mathbf{P = 66.26 \text{ kN}}$$
Ans.

10.6 APPLICATION TO LADDER PROBLEMS

A ladder resting against a wall is a typical case of friction problems in non-concurrent system of forces. Hence we have three equations of equilibrium available. From law of friction we have the equation $\frac{F}{N} = \mu$. Using equilibrium equations and friction law the problems can be solved. The procedure is illustrated with the examples below:

Example 10.10 A ladder of length 4 m, weighing 200 N is placed against a vertical wall as shown in Fig. 10.14(a). The coefficient of friction between the wall and the ladder is 0.2 and that between floor and the ladder is 0.3. The ladder, in addition to its own weight, has to support a man weighing 600 N at a distance of 3 m from *A*. Calculate the minimum horizontal force to be applied at *A* to prevent slipping.

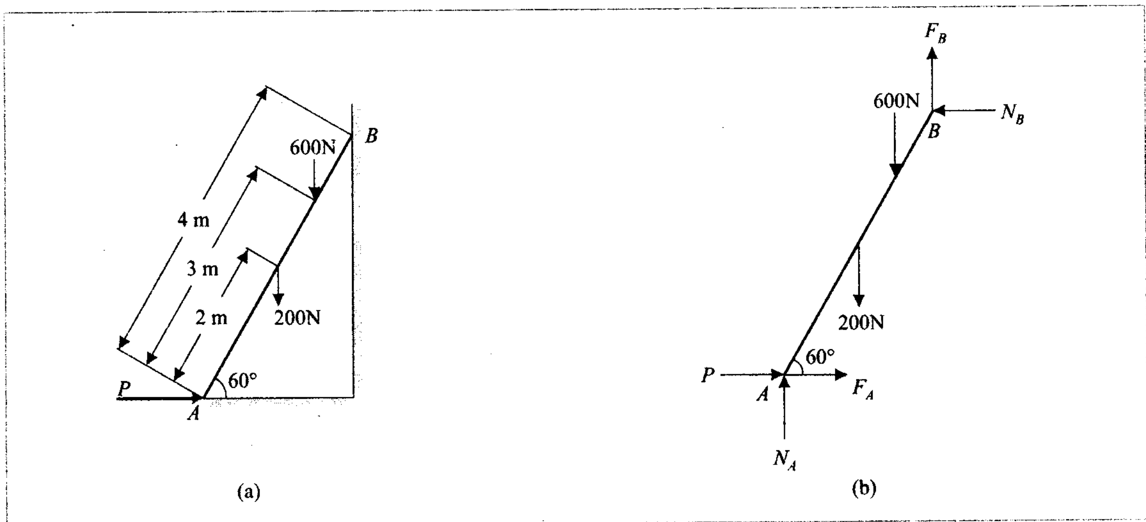


Fig. 10.14

Solution. The free body diagram of the ladder is as shown in Fig. 10.14(b).

$\sum M_A = 0 \rightarrow$

$$200 \times 2 \cos 60 + 600 \times 3 \cos 60 - F_B \times 4 \cos 60 - N_B \times 4 \sin 60 = 0$$

Dividing throughout by 4 and rearranging the terms, we get

$$0.866\, N_B + 0.5\, F_B = 275$$

From law of friction, $F_B = \mu N_B = 0.2\, N_B$

$\therefore$ $0.866\, N_B + 0.5 \times 0.2\, N_B = 275$

or $N_B = 284.7$ newton.

$\therefore$ $F_B = 56.94$ newton.

$\sum F_V = 0 \rightarrow$

$$N_A - 200 - 600 + F_B = 0$$

$$N_A = 743.06 \text{ newton}, \quad \text{since } F_B = 56.94$$

$\therefore$

$$F_A = \mu_A N_A$$

$$= 0.3 \times 743.06 = 222.9 \text{ newton}$$

$\sum F_H = 0 \rightarrow$

$$P + F_A - N_B = 0$$

$\therefore$

$$P = N_B - F_A = 284.7 - 222.9$$

i.e., $\quad$ **P = 61.8 newton** **Ans.**

Example 10.11 The ladder shown in Fig. 10.15(a) is 6m long and is supported by a horizontal floor and a vertical wall. The coefficient of friction between the floor and the ladder is 0.25 and between the wall and the ladder is 0.4. The weight of the ladder is 200 N and may be considered as a concentrated load at G. The ladder supports a vertical load of 900 N at C which is at a distance of 1 m from B. Determine the least value of α at which the ladder may be placed without slipping. Determine the reaction at that stage.

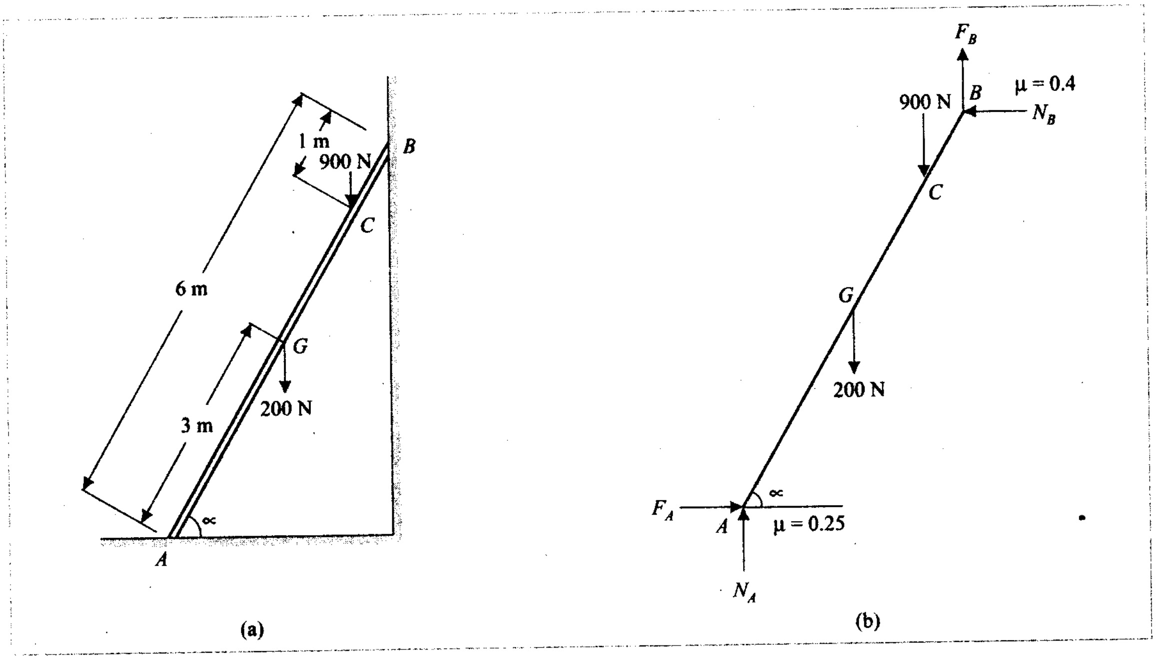


Fig. 10.15

Solution. Figure. 10.15(b) shows the free body diagram of the ladder.

From law of friction,

$$F_A = 0.25\, N_A \quad \text{...(i)}$$

$$F_B = 0.40\, N_B \qquad \text{...(ii)}$$

$$\Sigma F_V = 0 \rightarrow$$

$$N_A - 200 - 900 + F_B = 0$$

i.e., $$N_A + 0.4\, N_B = 900 + 200 = 1100 \qquad \text{...(iii)}$$

$$\sum F_H = 0 \rightarrow$$

$F_A - N_B = 0$ i.e., $F_A = N_B$

i.e., $0.25\, N_A = N_B$ ...(iv)

From eqns. (iii) and (iv), we get

$$N_A\,(1 + 0.4 \times 0.25) = 1100$$

or $\mathbf{N_A = 1000}$ **newton** **Ans.**

∴ $F_A = 0.25 \times N_A = 0.25 \times 1000 = \mathbf{250\ N}$ **Ans.**

From eqn. (iv) $N_B = 0.25\, N_A = \mathbf{250\ N}$ **Ans.**

∴ $F_B = 0.4 \times N_B = 0.4 \times 250 = \mathbf{100\ N}$ **Ans.**

$$\sum M_A = 0 \rightarrow$$

$$200 \times 3 \cos\alpha + 900 \times 5 \cos\alpha - F_B \times 6 \cos\alpha - N_B \times 6 \sin\alpha = 0$$

∴ Substituting the values of F_B and N_B, we get

$$200 \times 3 \cos\alpha + 900 \times 5 \cos\alpha - 100 \times 6 \cos\alpha - 250 \times 6 \sin\alpha = 0.$$

or $$4500 \cos\alpha = 1500 \sin\alpha$$

or $$\tan\alpha = 3$$

∴ $\alpha = \mathbf{71.57°}$ **Ans.**

Important Definitions

1. The maximum value of the frictional force, which develops between the two contacting surfaces when the motion is impending, is called *Limiting friction.*
2. Magnitude of limiting friction bears a constant ratio to the normal reaction. This ratio is called the *Coefficient of friction* $\left[\mu = \frac{F_{lim}}{N}\right]$.
3. The angle made by the resultant reaction (resultant of normal and limiting frictions) with normal to the contacting surfaces is known as *Angle of friction* $\left[\theta = \tan^{-1}\frac{F_{lim}}{N}\right]$.